Theory of Distributions

XIA Yusen

Fall 2021

HKUST

HW Grader: Yang YaHong. yyangct@connect.ust.hk.

Def. Principal value. $P.V.\frac{1}{x}$ is a distribution $\left(P.V.\frac{1}{x}\right)(\varphi) = \lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}dx \quad \forall \varphi\in C_c^\infty(\mathbb{R})$.

Prop. $\left(PV\frac{1}{x}\right)(\varphi) = \int \log|x|\,\varphi'(x)\,dx$, $\forall\varphi\in C_c^\infty(\mathbb{R})$.

Hint. $\int_{|x|\geq\varepsilon}\frac{\varphi}{x}dx = \int_\varepsilon^\infty \frac{\varphi}{x}dx + \int_{-\infty}^{-\varepsilon}\frac{\varphi}{x}dx$

$y=-x$, $dx=-dy$

$$= \int_\varepsilon^\infty \frac{\varphi}{x}dx + \int_\infty^\varepsilon \frac{\varphi(-y)}{-y}(-dy)$$

$$= \int_\varepsilon^\infty \frac{\varphi(x)-\varphi(-x)}{x}dx$$

MVT

$$= \int_\varepsilon^\infty \frac{x\cdot\psi(x)}{x}dx \quad \text{where } \psi(x)\in C_c^\infty$$

procedure to define a singular distribution called a *principal value distribution*, denoted by p.v.$(1/x)$. We define its action on a test function $\varphi\in\mathcal{S}(\mathbb{R})$ by

$$\text{p.v.}\frac{1}{x}(\varphi) = \lim_{\epsilon\to0^+}\int_{|x|>\epsilon}\frac{\varphi(x)}{x}\,dx.$$

The limit is finite because of a cancellation between the nonintegrable contributions of $1/x$ for $x<0$ and $x>0$:

$$\text{p.v.}\frac{1}{x}(\varphi) = \lim_{\epsilon\to0^+}\int_\epsilon^\infty\frac{\varphi(x)-\varphi(-x)}{x}\,dx = \int_0^\infty\frac{\varphi(x)-\varphi(-x)}{x}\,dx.$$

The integrand is bounded at $x=0$ since φ is smooth. For $x>0$, we have

$$\left|\frac{\varphi(x)-\varphi(-x)}{x}\right| \leq \frac{1}{x}\int_{-x}^{x}|\varphi'(t)|\,dt \leq 2\|\varphi'\|_\infty,$$

so the continuity of p.v.$(1/x)$ on $\mathcal{S}$ follows from the estimate

$$\begin{aligned}\left|\text{p.v.}\frac{1}{x}(\varphi)\right| &\leq \int_0^1\left|\frac{\varphi(x)-\varphi(-x)}{x}\right|dx + \int_1^\infty\left|\frac{x[\varphi(x)-\varphi(-x)]}{x^2}\right|dx\\ &\leq 2\|\varphi'\|_\infty + 2\|x\varphi\|_\infty\\ &= 2\left(\|\varphi\|_{0,1}+\|\varphi\|_{1,0}\right).\end{aligned}$$

Def. Derivative of a distribution. if $u\in\mathcal{D}'(\mathbb{R}^n)$, define $\frac{\partial}{\partial x_j}u$ to be:

$$\left(\frac{\partial}{\partial x_j}u\right)(\varphi) = -u\left(\frac{\partial\varphi}{\partial x_j}\right) \quad \forall\varphi\in C_c^\infty(\mathbb{R}^n)$$

Prop. $\frac{\partial}{\partial x_j}u\in\mathcal{D}'(\mathbb{R}^n)$ if $u\in\mathcal{D}'(\mathbb{R}^n)$.

Similarly, we can define $D^\alpha u\in\mathcal{D}'(\mathbb{R}^n)$ as: $(D^\alpha u)(\varphi) := (-1)^{|\alpha|}u(D^\alpha\varphi)$

One can differentiate as much as he wants.

Example. Heaviside function $H(x) = \begin{cases}1 & x>0\\ 0 & x<0\end{cases} \in L^1_{loc}(\mathbb{R})$

$$T_H(\varphi) := \int_{-\infty}^{+\infty}H(x)\varphi(x)\,dx$$

$$\left(\frac{d}{dx}T_H\right)(\varphi) = -T_H(\varphi'(x)) = -\int_{-\infty}^{+\infty}H(x)\varphi'(x)\,dx = -\int_0^{+\infty}H\varphi'\,dx = \varphi(0)$$

$\Rightarrow \delta_0 = \frac{d}{dx}T_H$ in $\mathcal{D}'(\mathbb{R})$

Example: $f=\log|x|\in L^1_{loc}$ but $f'=\frac{1}{x}\notin L^1_{loc}$

$$T_f(\varphi) = \int_{-\infty}^{+\infty}\log|x|\,\varphi(x)\,dx, \quad \varphi\in C_c^\infty(\mathbb{R})$$

$$\left(\frac{d}{dx}T_f\right)(\varphi) = -T_f(\varphi') = -\int\log|x|\,\varphi'(x)\,dx = \left(P.V.\frac{1}{x}\right)(\varphi) = \lim_{\varepsilon\to0^+}\int_{|x|\geq\varepsilon}\frac{\varphi}{x}dx$$

Now look at 1D wave equation $\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0$. $u \in C^2(\mathbb{R} \times \mathbb{R})$

General sol. $u(x,t) = f(x+ct) + g(x-ct)$, $f, g \in C^2(\mathbb{R})$

Now what if $f, g \in \mathcal{D}'(\mathbb{R})$? How to define $f(x+ct)$, $g(x-ct)$?

Assume $f, g \in L^1_{loc}(\mathbb{R})$. take $\varphi \in C_c^\infty(\mathbb{R})$ and test it against $T_{f(x+ct)}$:

$$\int_{\mathbb{R}^2} f(x+ct)\, \varphi(x,t)\, dx\, dt = \int_{\mathbb{R}^2} f(y)\, \varphi(y-ct, t)\, dy\, dt \qquad y := x+ct$$

Hence define $f(x+ct)$ as a distribution : $f(x+ct)(\varphi) = \int_{\mathbb{R}^2} f(y)\, \varphi(y-ct, t)\, dy\, dt$

Similarly, $g(x-ct)(\varphi) = \int_{\mathbb{R}^2} g(y)\, \varphi(y+ct, t)\, dy\, dt$ as a distribution.

$u_0 \in \mathcal{D}'(\mathbb{R})$. $u_0(x \pm ct) \in \mathcal{D}'(\mathbb{R}^2)$ acting on functions of 2 variable. defined as follows: take $\varphi(x,t) \in C_c^\infty(\mathbb{R}^2)$

output a function of t

$$u_0(x \pm ct)(\varphi) = \int \boxed{u_0\big(\varphi(\cdot \mp ct, t)\big)}\, dt$$

u_0 acting on a function of x, one variable

the "dot" variable

$\varphi(\cdot \mp ct, t) \in C_c^\infty(\mathbb{R}_x)$ a function of x

$\varphi(\cdot \mp ct, t)(x) = \varphi(x \mp ct, t)$, $x \in \mathbb{R}$

Claim: $h(t) := u_0(\varphi(\cdot \mp ct, t)) \in C_c(\mathbb{R}_t) \Rightarrow$ the integral above makes sense.

Proof: For large t, $\varphi(\cdot \mp ct, t) = 0$ since $\varphi \in C_c^\infty(\mathbb{R}_x, \mathbb{R}_t)$

Let $t_j \to t_0$. WANT: $u_0(\varphi(\cdot \mp ct_j, t_j)) \to u_0(\varphi(\cdot \mp ct_0, t_0))$ as $j \to \infty$.

$\varphi(\cdot \mp ct_j, t_j) \to \varphi(\cdot \mp ct_0, t_0)$ in $C_c^\infty(\mathbb{R})$ since φ is smooth.

Then by continuity of u_0, we are done.

Prop. $u \in \mathcal{D}'(\mathbb{R}^2)$ $\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$. Factorization holds for distributions.

Proof: $\left(\frac{\partial}{\partial x}\frac{\partial}{\partial t} u\right)(\varphi) = -\left(\frac{\partial}{\partial t} u\right)\left(\frac{\partial \varphi}{\partial x}\right)$

$= u\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\varphi\right)$ $\varphi \in C_c^\infty$ Hence switch the order of differentiation.

$= \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x} u\right)(\varphi)$

$\Rightarrow$ Mixed partial derivative then holds for distributions.

$\Rightarrow$ It implies the prop.

Prop. $u_0 \in \mathcal{D}'(\mathbb{R})$ $\quad u_0(x \pm ct)(\varphi) := \int u_0(\varphi(\cdot \mp ct, t))\,dt$ solves the equation

$$\left(\frac{\partial}{\partial t} \mp c\frac{\partial}{\partial x}\right) u_0(x \pm ct)(\varphi) = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^2)$$

RMK. $u_0(x \pm ct)$ then solves the wave equation $u_{tt} = c^2 u_{xx}$

Proof. Just prove "−" case:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u_0(x-ct)(\varphi) = -u_0(x-ct)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(\varphi)$$

$$= -\int u_0\left[\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(\varphi(\cdot + ct, t))\right] dt$$

$$= -\int u_0\left(\frac{d}{dt}\varphi(\cdot + ct, t)\right) dt$$

$$\overset{?}{=} -\int \frac{d}{dt}\left[u_0(\varphi(\cdot + ct, t))\right] dt$$

$$= u_0(\varphi(\cdot + ct, t))\Big|_{-\infty}^{+\infty} = 0 \quad \text{since compactly supported.}$$

$$\frac{d}{dt}\underbrace{u_0(\varphi(\cdot + ct, t))}_{h(t)} \overset{\text{linearity continuous}}{=} \lim_{h\to 0} u_0\left(\frac{\varphi(\cdot - c(t+h), t+h) - \varphi(\cdot - ct, t)}{h}\right)$$

$$= u_0\left(\lim_{h \to 0} \frac{\varphi(\cdot, t+h) - \varphi(t)}{h}\right)$$

$$= u_0\left(\frac{d}{dt}\varphi(\cdot)\right) \quad \text{since } \varphi \in C_c^\infty$$

Example. $\delta_0(x \pm ct)(\varphi) = \int \delta_0(\varphi(\cdot \mp t, t))\,dt$

$$= \int \varphi(\mp ct, t)\,dt \quad \varphi \in C_c^\infty(\mathbb{R}^2)$$

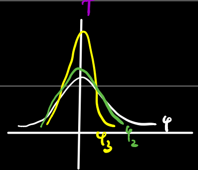
Notation. $T \in \mathcal{D}'$, $\varphi \in C_c^\infty$. $\quad \langle T, \varphi \rangle := T(\varphi)$

Convergence of distribution.

Let $u_j \in \mathcal{D}'$, $u \in \mathcal{D}'$. We say $u_j \to u$ in $\mathcal{D}'$ if $u_j(\varphi) \to u(\varphi)$ $\forall \varphi$ in C_c^∞. Weak convergence.
conv. in $\mathbb{C}$.

Example (Approximation of the identity) take $\varphi \in C_c^\infty(\mathbb{R}^n)$ $\int_{\mathbb{R}^n} \varphi\,dx = 1$

$\varphi_j(x) = j^n \varphi(jx)$



Then $\int \varphi_j(x)\,dx = 1$. And $T_{\varphi_j}(\psi) = \int \varphi_j \psi \, dx \to \delta_0(\psi) = \psi(0) \quad \forall \psi \in C_c^\infty(\mathbb{R}^n)$.

Hence $\varphi_j \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$.

Multiplication by C_c^∞: $u \in \mathcal{D}'$. $f \in C_c^\infty$. define fu as:

$$(fu)(\varphi) = u(f\varphi).$$

Example: $f(x) = x$. $u = \delta_0$. $(f\delta_0)(\varphi) \overset{\text{def}}{=} \delta_0(f\varphi) = f(0)\varphi(0) = 0$. Here f, δ_0 both not zero but $f\delta_0 \equiv 0$.

Theorem. $u \in \mathcal{D}'(X) \iff$

① $u : C_c^\infty(X) \to \mathbb{C}$ is linear

(*) ② $\forall\, K \subset X$ compact, $\exists\, C_K$, $N_K \in \mathbb{N}$.

$$|u(\varphi)| \le C_K \sum_{|\alpha| \le N_K} \sup_{x \in K} |\varphi^{(\alpha)}(x)|$$

$\forall\, \varphi \in C_c^\infty(K)$

Proof: $\Leftarrow$: Suppose ① and ② holds.

Then for any converging sequence $\{\varphi_n\} \subseteq C_c^\infty(X)$,
$\varphi_n \to 0$.

$\iff \exists\, K \subseteq X$ compact. $\|\partial^\alpha \varphi_j(x)\|_\infty \to 0 \quad \forall \alpha$. on K.

$$|u(\varphi_j) - 0| = |u(\varphi_j)| \leq C_K \sum_{|\alpha|\leq N_K} \|\partial^\alpha \varphi_j\|_\infty$$

$$\to 0 \quad \text{as } j \to \infty.$$

Hence u is continuous; $u \in \mathcal{D}'(X)$

$\Rightarrow$: Suppose now $u \in \mathcal{D}'$: u is linear.

proof by contradiction, Assume ② is not true.

$\exists K \subseteq X$ compact. $\forall N \in \mathbb{N}$:
$\exists \varphi_N \in C_c^\infty(K)$.

$$|u(\varphi_N)| > N \sum_{|\alpha|\leq N} \|\partial^\alpha \varphi_N\|_\infty.$$

Take $\psi_N = \dfrac{\varphi_N}{\sum_{|\alpha|\leq N} \|\partial^\alpha \varphi_N\|_\infty} \in C_c^\infty(K).$

$$|u(\psi_N)| > N.$$

$$\Leftrightarrow \left|u\left(\frac{\psi_N}{N}\right)\right| > 1.$$

Claim: $\left(\frac{\psi_N}{N}\right)_{N=1}^\infty \to 0$ in $C_c^\infty(K)$.

proof: $\frac{\psi_N}{N} = \frac{\varphi_N}{N \sum_{|\alpha|\le N} \|\partial^\alpha \varphi_N\|_\infty}$.

$$\left|\partial^\beta\left(\frac{\psi_N}{N}\right)\right| = \frac{|\partial^\beta \varphi_N|}{N \sum_{|\alpha|\le N} \|\partial^\alpha \varphi_N\|_\infty} \le \frac{1}{N}$$

as long as $|\beta| \le N$.

$\Rightarrow \quad \sup_K \left|\partial^\beta\left(\frac{\psi_N}{N}\right)\right| \le \frac{1}{N}$. if $|\beta| \le N$

$\Rightarrow \quad \frac{\psi_N}{N} \to 0$ in $C_c^\infty(K)$.

However. $u\left(\frac{\psi_N}{N}\right) \ge 1$.

(Contradiction to the fact $u \in \mathcal{D}'(X)$.

$\square$

Definition. Order of distribution

min N s.t. $(*)$ is valid for any compact set $K \subseteq X$.

(Might be infinity)

Example. 1) Order of $\delta(x)$:

$$\delta(\varphi) = \varphi(0)$$

order $= 0$.

2) p.v. $\frac{1}{x}$: $\quad p.v.\,\frac{1}{x}(\varphi) = -\int \varphi' \log|x|\,dx$

order $= 1$.

Exercise: Find a distribution of order ∞.

Hint: $\sum_{N=0}^{\infty} \delta_N^{(N)}$.

Definition. Restriction of distribution.

$X \subseteq \mathbb{R}^n$ open. $V \subseteq X$ open.

$u \in \mathcal{D}'(X)$.

$u|_V \in \mathcal{D}'(V)$ is a distribution on V

$$u|_V(\varphi) := u(\varphi)$$

$$\forall \varphi \in C_c^\infty(V)$$

Definition $X \subseteq \mathbb{R}^n$ open

$V \subseteq X$ open.

$u = 0$ on V if $u(\varphi) = 0 \ \forall \varphi \in C_c^\infty(V)$

proposition $X_i \subseteq X$ open. $i = 1, 2, \cdots, N$

$u = 0$ on $X_i \ \forall i$

Then $u = 0$ on $\bigcup_{i=1}^N X_i$

proof: Partition of Unity: $X := \bigcup_{i=1}^N X_i$

$K \subseteq X$ compact.

then $\exists \chi_i \in C_c^\infty(X_i)$. $0 \leq \chi_i \leq 1$

$\sum \chi_i \equiv 1$ on a neighbourhood of K.

Now $\varphi \in C_c^\infty(\bigcup_{i=1}^N X_i)$.

Suppose $K := \operatorname{supp} \varphi$.

$$\varphi = \sum_{i=1}^N \underbrace{(\chi_i \varphi)}_{\in C_c^\infty(X_i)}$$

$$u(\varphi) = \sum_{i=1}^N u(\varphi \chi_i) \underset{u=0 \text{ on } X_i}{=} \sum_{i=1}^N 0 = 0. \quad \square$$

Definition.
Support of distribution

$\operatorname{supp} u :=$ complement of the largest open set where $u=0$.

existence: $V = \bigcup_{\alpha\in A} X_\alpha$ where $u=0$ on X_α.

prove $u=0$ on V: $\forall \varphi \in C_c^\infty(V)$.

$K := \operatorname{supp} \varphi$.

$\exists$ Finite number $X_{\alpha_1}, \dots, X_{\alpha_N}$ covers K.

Apply the previous proposition. Done.

Example.

1) $\delta(x)$: $\operatorname{supp} \delta = \{0\}$ since

$$\forall \varphi \in C_c^\infty(\mathbb{R}\setminus\{0\}),\ \delta(\varphi)=0.$$

2) $\operatorname{supp}\left(p.v.\frac{1}{x}\right) = \mathbb{R}$

$$p.v.\frac{1}{x}(\varphi) = \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{\varphi}{x}\,dx$$

$$= -\int \log|x| \cdot \varphi'(x)\,dx.$$

On $\mathbb{R}\setminus\{0\}$: $pv\frac{1}{x} = \frac{1}{x}$.

$$\operatorname{supp}\frac{1}{x} = \overline{\mathbb{R}\setminus\{0\}} = \mathbb{R}$$

$\Rightarrow \operatorname{supp}(pv\frac{1}{x}) = \mathbb{R}$.

Definition $\mathcal{E}'(X) := \{u \in \mathcal{D}'(X) \mid \operatorname{supp} u \text{ is compact}\}$

is the space of compactly supported distributions.

Example 1) $\delta(x)$ supported at $\{0\}$.

Theorem Given $u \in \mathcal{D}'(X)$.

$u \in \mathcal{E}'(X)$ iff $\exists K \subseteq X$ compact.

$\exists C > 0$. $N \in \mathbb{N}$.

$$(**) \quad |u(\varphi)| \leq C \sum_{|\alpha| \leq N} \sup_K \|\partial^\alpha \varphi\|.$$

$\forall \varphi \in C_c^\infty(X)$

Proof of Theorem:

$\Leftarrow$: take $\varphi \in C_c^\infty(X)$

$\operatorname{supp} \varphi \cap K = \phi$.

$u(\varphi) = 0$ by $(**)$

$\Rightarrow \operatorname{supp} u \subseteq K$.

$\rightarrow$ u is compactly supported.

$\Rightarrow$: $u \in \mathcal{D}'(X)$ has compact support.

$\operatorname{supp} u = K \subseteq X$ compact.

By (*), $|u(\varphi)| \le C \sum_{|\alpha| \le N} \sup_K |\partial^\alpha \varphi|$

$\forall \varphi \in C_c^\infty(K)$.

Take a bump function f s.t.

$f \equiv 1$ on a nbhd of K.

Let $\psi \in C_c^\infty(X)$.

$$\psi = f\psi + (1-f)\psi$$

$$u(\psi) = u(f\psi) + u((1-f)\psi)$$

$(1-f)\psi = 0$ on a nbhd of K

Hence $u((1-f)\psi) = 0$. since $\operatorname{supp} u = K$.

$\Rightarrow u(\psi) = u(f\psi)$

Now $\operatorname{supp} f\psi = \tilde{K}$ compact

Apply (*) on $\tilde{K}$:

$$|u(\psi)| = |u(f\psi)| \le C_{\tilde{K}} \sum_{|\alpha| \le N} \sup_{\tilde{K}} |\partial^\alpha (f\psi)|$$

$$\left|\partial^\alpha(f\psi)\right| = \left|\sum_\beta C_{\alpha,\beta}\, \partial^\beta f \cdot \partial^{\alpha-\beta}\psi\right|$$

$$\lesssim \sum_{|\alpha|\le N} \sup_X |\partial^\alpha \psi|.$$

$$\forall \psi \in C_c^\infty(X)$$

(**) is shown.

$\square$

(**) $u \in \mathcal{E}'(X)$ $\quad |u(\varphi)| \le C \sum_{|\alpha|\le N} \sup_K |\partial^\alpha \varphi|$

for some $C>0$, $N\in\mathbb{N}$, $K\subseteq X$ compact.

– any $\varphi \in C_c^\infty(X)$.

Corollary The order of $u \in \mathcal{E}'$ is finite

Definition. Convergence in $C^\infty(X)$:

$\{f_j\}\subseteq C^\infty(X)$ $\quad f_j \to 0$ in $C^\infty(X)$

iff $\forall K \subseteq X$ compact,

$$\sup_K |\partial^\alpha f_j| \to 0$$

$\forall$ multi-index α.

Definition. $u : C^\infty(X) \to \mathbb{C}$ linear is continuous if $\forall \{f_j\} \subseteq C^\infty(X)$ $f_j \to 0$ in $C^\infty(X)$ then $u(f_j) \to 0$ in $\mathbb{C}$.

Theorem $u \in \mathcal{E}'(X)$ then u has an extension $\tilde{u} : C^\infty(X) \to \mathbb{C}$ linear, continuous

proof. Take $f \in C^\infty(X)$. $\operatorname{supp} u = K$ compact.

Let $\psi \in C_c^\infty(X)$ $\psi \equiv 1$ on a nbhd of K.

Then $f\psi \in C_c^\infty(X)$, $f\psi \equiv f$ on K

$$f = \psi f + (1-\psi) f$$

Define $\tilde{u}(f) := u(\psi f)$.

Check def is independent of ψ:

Let $\tilde{\psi}$ be another bump function s.t $\tilde{\psi} \equiv 1$ on a nbhd of K

$u((\psi - \tilde{\psi})f) = 0$ since $\tilde{\psi} f = \psi f$ on a nbhd of K.

Check continuity of extended $\tilde{u}$:

As $\{f_j\} \in C^\infty$ $f_j \to 0$: $\psi f_j \to 0$ in $C_c^\infty(X)$

Since $\mathrm{supp}\, \psi f_j \subseteq \mathrm{supp}\, \psi = K$.

$$\partial^\alpha(\psi f_j) := \cdots \quad \text{Libniz rule.}$$

$$\to 0 \text{ uniformly}$$

$$\tilde{u}(f_j) = u(\psi f_j) \xrightarrow[\uparrow]{} 0$$

continuity of u

$\square$

Summary $\quad u \in \mathcal{D}' \iff u : C_c^\infty \to \mathbb{C}$ satisfies $(*)$

$u \in \mathcal{E}' \iff u : C^\infty \to \mathbb{C}$ satisfies $(**)$

However, multiplication of distribution could be a problem.

proposition. δ^2 can not be defined in $\mathcal{D}'$ in the sense that if $\varphi_j \to \delta$,

then $\varphi_j^2 \to \delta^2$.

proof. Take $\varphi \in C_c^\infty$ with $\int_{\mathbb{R}^n} \varphi = 1$

$$\varphi_j(x) := j^n \varphi(jx).$$

$\int \varphi_j = 1 \quad \forall j. \qquad \varphi_j \to \delta$ in $\mathcal{D}'$

What about φ_j^2 ?

$$\varphi_j^2 = j^{2n} \varphi(jx)^2$$

Take $\psi \in C_c^\infty(\mathbb{R}^n)$

$$\int \psi \varphi_j^2 \, dx = \int j^{2n} \psi \varphi^2(jx) \, dx$$

$y = jx$

$dy = j^n dx$

$$= \underbrace{j^n}_{\to \infty} \underbrace{\int \psi\left(\frac{y}{j}\right) \varphi^2(y) \, dy}_{\to \psi(0) \int \varphi^2} \to \infty \text{ as } j \to \infty.$$

$\square$

Theorem Suppose $u \in \mathcal{D}'(X)$. $x_0 \in X$
$\operatorname{supp} u = \{x_0\}$, then $\exists\, N \in \mathbb{N}$.

$$u = \sum_{|\alpha| \le N} a_\alpha \partial^\alpha \delta_{x_0}, \quad a_\alpha \in \mathbb{C}.$$

proof: WLOG. Suppose $x_0 = 0$. $X = \mathbb{R}$.

Since u is compactly supported,

order $u = N < +\infty$.

take $\varphi \in C_c^\infty(\mathbb{R})$

$$\varphi = \varphi(0) + \varphi'(0)x + \frac{x^2}{2}\varphi''(0) + \cdots + \frac{x^N}{N!}\varphi^{(N)}(0) + x^{N+1}\underbrace{R_{N+1}(x)}_{\in C^\infty(\mathbb{R})}$$

Take $\psi \in C_c^\infty(\mathbb{R})$ s.t. $\psi \equiv 1$ on $|x| \le \frac{1}{2}$
$\psi \equiv 0$ on $|x| \ge 1$

$$f_\varepsilon(x) := \varphi(x)\,\psi\left(\frac{x}{\varepsilon}\right)$$

$f_\varepsilon \equiv 0$ on $|x| \ge \varepsilon$.

Write
$$f_\varepsilon(x) = f_\varepsilon(0) + f_\varepsilon'(0)x + \cdots + \frac{f_\varepsilon^{(N)}}{N!}x^N + R_{n+1}(x)x^{N+1}$$
$$= \varphi(0) + \varphi'(0)x + \cdots + \frac{\varphi^{(N)}(0)}{N!}x^N + x^{N+1}\tilde{R}_{N+1}(x)$$

where $\tilde{R}_{N+1}(x,\varepsilon) = \left(\varphi(\eta)\psi\left(\frac{1}{\varepsilon}\right)\right)^{(N+1)}$

$0 < \eta < x$. (MVT). Langrange remainder.

$u(f_\varepsilon) = u(\psi(\frac{x}{\varepsilon})\varphi) = u(\varphi) \quad \forall \varepsilon > 0$

since $\operatorname{supp} u = \{0\}$

$u(\varphi) = \varphi(0)\underbrace{u(1)}_{\text{u applied to const function } 1} + \varphi'(0)u(x) + \dots + \frac{\varphi^{(N)}(0)}{N!}u(x^{N+1})$

$+ u\left(x^{N+1}\tilde{R}_{N+1}(x)\right)$

$= c_0\delta_0(\varphi) + c_1\delta_0'(\varphi) + \dots + c_N\delta_0^{(N)}(\varphi)$

$+ \underbrace{u\left(x^{N+1}\tilde{R}_{N+1}(x,\varepsilon)\right)}_{\text{want it to be zero.}}$

$\varphi = \varphi\psi(\frac{x}{\varepsilon}) + \varphi(\underbrace{1-\psi(\frac{x}{\varepsilon})}_{=0 \text{ near zero}})$

$u(\varphi) = u(\varphi\psi(\frac{x}{\varepsilon})) + \underbrace{\cancel{u(\varphi(1-\psi(\frac{x}{\varepsilon})))}}_{=0 \text{ since } \operatorname{supp} u = \{0\}}$

$\forall \varepsilon > 0$.

Now use $\operatorname{supp} u$ compact again:

$$u(f) \le C \sum_{|\alpha|\le N} \sup_{K} |\partial^\alpha f| \quad f \in C_c^\infty(\mathbb{R})$$

$K = [-1, 1]$. ($\forall\, K \supseteq \{0\}$ would be fine)

$$u\left(x^{N+1} \tilde{R}_{N+1}(x,\varepsilon) \right) = u\left[x^{N+1} \left(f_\varepsilon \right)^{(N+1)} \Big|_{x=\eta} \right]$$

Claim: $\left| \partial^\alpha \left(x^{N+1} \underbrace{f_\varepsilon}_{\operatorname{supp} \subseteq \{ |x| \le \varepsilon \}} \right) \right| \le C\varepsilon \quad |\alpha| \le N$.

proof of the claim: $\alpha = 0$: $\partial^0 (x^{N+1} f_\varepsilon)$

$$= x^{N+1} \psi\left(\tfrac{x}{\varepsilon}\right) \varphi(x)$$

$$\sup_{[-1,1]} \left| x^{N+1} \psi\left(\tfrac{x}{\varepsilon}\right) \varphi(x) \right| \le C \varepsilon^{N+1}$$

$\alpha = 1$: $(x^{N+1} f_\varepsilon)' = f_\varepsilon' x^{N+1} + (N+1) f_\varepsilon x^N$.

$$\sup |(x^{N+1} f_\varepsilon)'| \le \sup |f_\varepsilon'| \cdot \varepsilon^{N+1} + (N+1) \sup |f_\varepsilon| \, \varepsilon^N.$$

$$\lesssim C \varepsilon^N$$

The rest can be shown by induction. $(x^{N+1} f_\varepsilon)^{(\alpha)} \lesssim \varepsilon^{N+1-\alpha}$

Now $u(x^{N+1} f_\varepsilon) \le C\varepsilon$

Take $\varepsilon \to 0^+$, we are done. $\square$

Convolutions. $f \in C_c^\infty(\mathbb{R}^n)$. $g \in L^1_{loc}(\mathbb{R}^n)$

$$(f*g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Extendible to $g \in \mathcal{D}'(\mathbb{R}^n)$.

Proposition: 1) $(f*g)(x) = (g*f)(x)$.

2) $\operatorname{supp} f*g \subseteq \operatorname{supp} f + \operatorname{supp} g$.

Proofs are omitted.

Theorem(*) (Mollifier) $f \in C_c^\infty(\mathbb{R}^n)$, $g \in L^1_{loc}(\mathbb{R}^n)$

$\Rightarrow f*g \in C^\infty(\mathbb{R}^n)$. $\partial^\alpha(f*g) = \partial^\alpha f * g$

$= f * \partial^\alpha g$.

proof of the theorem. 1) $f*g$ is continuous:

Suppose $x_j \to x_0$.

$$(f*g)(x_j) = \int f(x_j-y)\, g(y)\, dy$$

$$\xrightarrow{\text{LDCT}} \int f(x-y) g(y)\, dy$$

since $f(x_j-y)\,g(y)$ is integrable.

$$= (f*g)(x).$$

2) differentiable: Take $x_0 \in \mathbb{R}^n$, $h_j \to 0$

Difference quotient in the e_k-direction.

$$\frac{(f*g)(x_0+h_j e_k) - f*g(x_0)}{h_j}$$

$$= \int \frac{f(x_0+h_j e_k - y) - f(x_0-y)}{h_j} g(y)\,dy$$

LDCT

since $f'(x-y)g(y)$ is integrable.

$$= \int \frac{\partial f}{\partial x_k}(x-y)\, g(y)\,dy$$

In the same way, one can show

$$f*g \in C^\infty.$$

$\square$

We now extend convolution to distributions:

Definition: Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $f \in C_c^\infty(\mathbb{R}^n)$

$$(u*f)(x) := u(f(x-\cdot)(y))$$

as a function of y

$*: \mathcal{D}' \times C_c^\infty \to C^\infty$.

Example $(\delta * f)(x) := \delta(f(x-\cdot)(y))$
$= f(x-0) = f(x)$

Hence δ is the identity of convolution operation.

proposition: $u \in \mathcal{D}'(\mathbb{R}^n)$ $f \in C_c^\infty(\mathbb{R}^n)$ then
$u * f \in C^\infty(\mathbb{R}^n)$, $\partial^\alpha(u*f) = \partial^\alpha u * f$
$= u * \partial^\alpha f$

proof of the proposition: 1) Continuity: take $x_j \to x_0$

$(u*f)(x_j) = u(f(x_j - \cdot)(y))$

But $f(x_j - \cdot)(y) \to f(x_0 - \cdot)(y)$ in $C_c^\infty(\mathbb{R}^n)$

Hence by continuity of $u \in \mathcal{D}'(\mathbb{R}^n)$,
we get $(u*f)(x_j) \to (u*f)(x_0)$

2) Differentiability:

$$\frac{(u*f)(x+h_j e_k) - (u*f)(x)}{h_j}$$

$$\overset{\text{linearity of } u}{=} u\left(\frac{f(x+h_j e_k - \cdot) - f(x-\cdot)}{h_j}\right)$$

Claim: $\dfrac{f(x+h_je_k-\cdot)-f(x-\cdot)}{h_j} \longrightarrow \dfrac{\partial f}{\partial x_k}(x)$ in $C_c^\infty(\mathbb{R}^n)$

The rest are similar. $\square$

Definition $u \in \mathcal{D}'(\mathbb{R}^n)$ $f \in C_c^\infty(\mathbb{R}^n)$

$$f * u := u * f.$$

Definition $u \in \mathcal{E}'(\mathbb{R}^n)$ $f \in C^\infty(\mathbb{R}^n)$

$$(u*f)(x) = (f*u)(x)$$

$$:= u\big(f(x-\cdot)(y)\big)$$

Exercise. suppose $u \in \mathcal{D}'(\mathbb{R}^n)$ $f \in C_c^\infty(\mathbb{R}^n)$

then $\operatorname{supp} u*f \subseteq \operatorname{supp} u + \operatorname{supp} f$

Convolution of two distributions.

Motivation: $u, v \in C_c^\infty$, $\varphi \in C_c^\infty$.

$$(u*v)(\varphi) := \iint u(x-y)\,v(y)\,dy\ \varphi(x)\,dx$$

$$= \iint u(y)\,v(x-y)\,dy\ \varphi(x)\,dx$$

$$= \int u(y) \underbrace{\int v(x-y)\,\varphi(x)\,dx}\ dy$$

$$\int v(x-y)\varphi(x)\,dx = \widetilde{v * \tilde{\varphi}}(y) \text{ where } \tilde{f}(x) = f(-x).$$

Indeed,
$$\begin{aligned}
\widetilde{v * \tilde{\varphi}}(y) &= v * \tilde{\varphi}(-y) \\
&= \int v(-y-x)\,\tilde{\varphi}(x)\,dx \\
&= \int v(-y-x)\,\varphi(-x)\,dx \\
(-x = z) \quad &= \int v(z-y)\,\varphi(z)\,dz.
\end{aligned}$$

$$\Rightarrow (u * v)(\varphi) = u\left(\widetilde{v * \tilde{\varphi}}\right)$$

This motivates the following definition:

Definition. $u \in \mathcal{D}'(\mathbb{R}^n)$ $v \in \mathcal{E}'(\mathbb{R}^n)$

$u * v = v * u$ s.t.

$$(u * v)(\varphi) := u(\underbrace{\widetilde{v * \tilde{\varphi}}}_{\in C_c^\infty}), \ \varphi \in C_c^\infty(\mathbb{R}^n)$$

Proposition. $u \in \mathcal{D}',\ v \in \mathcal{E}' \Rightarrow u * v \in \mathcal{D}'$.

proof: Linearity ✓

Continuity: $\varphi_j \to 0$ in $C_c^\infty(\mathbb{R}^n)$

supp $v \subseteq K$ compact

$\Rightarrow$ supp $\widetilde{v * \tilde{\varphi}_j} \subseteq K'$ compact.

$$\partial^\alpha \left(\widetilde{v * \tilde{\varphi}_j}\right) = \widetilde{v * \partial^\alpha \tilde{\varphi}_j}$$

$$= v\left(\partial^\alpha \tilde{\varphi}_j (x - \cdot)(y)\right) \xrightarrow{j \to \infty} 0$$

since v is continuous and $\varphi_j \to 0$ in $C_c^\infty(\mathbb{R}^n)$

$\Rightarrow u(\widetilde{v * \tilde{\varphi}_j}) \to 0.$

Definition. $u \in \mathcal{E}'(\mathbb{R}^n)$ $v \in \mathcal{D}'(\mathbb{R}^n)$

$$(u * v)(\varphi) := u(\widetilde{v * \tilde{\varphi}})$$

Example. $u * \delta$: $u * \delta(\varphi) = u(\widetilde{\delta * \tilde{\varphi}})$

$\overset{\delta * \varphi = \varphi}{=} u(\tilde{\tilde{\varphi}})$

$\forall \varphi \in C_c^\infty(\mathbb{R}^n)$ $= u(\varphi)$

$\Rightarrow \delta * u = u * \delta = u.$

Again, δ is the identity of convolution.

proposition. $C^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}'(\mathbb{R}^n)$

proof: Recall: $\varphi_j = j^n \varphi(jx)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$

$\int \varphi = 1$

$\Rightarrow \varphi_j \to \delta$ in $\mathcal{D}'$.

Take $\forall\, u \in \mathcal{D}'(\mathbb{R}^n)$.

$u = u * \delta$

$u_j := u * \varphi_j \in C^\infty(\mathbb{R}^n)$

To prove $u_j \to u$ in $\mathcal{D}'(\mathbb{R}^n)$, take $f \in \underbrace{C_c^\infty(\mathbb{R}^n)}$

$u_j(f) = (u * \varphi_j)(f) = u(\widetilde{\varphi_j * \tilde{f}})$

Claim: $\widetilde{\varphi_j * \tilde{f}} \to \widetilde{\delta * \tilde{f}} = f$ in C_c^∞.

proof of the claim: It suffices to show $\varphi_j * g \to g$.

$$\varphi_j * g = \int \varphi_j(x-y) g(y)\, dy$$

$$= \int \varphi_j(y)\, g(x-y)\, dy$$

$$|\varphi_j * g - g| \le \int |\varphi_j(y)|\, |g(x-y) - g(x)|\, dy$$

$$= \int |\varphi(z)|\, \underbrace{|g(x - \tfrac{z}{j}) - g(x)|}_{\to 0}\, dz$$

$$\le \sup_{x \in K} |\varphi| \cdot \int | \quad - \quad | dz \to 0.$$

Similarly:

$$\partial^\alpha(\varphi_j * g) = \varphi_j * \partial^\alpha g \to \partial^\alpha g$$

$u_j(f) \to u(f)$. $\square$

Done.