Fourier transform.

$f \in C_c^\infty$. $\hat{f}(\xi) = e^{-ix\cdot\xi} f(x)\,dx$, $\xi \in \mathbb{R}^n$.

But $C_c^\infty(\mathbb{R}^n)$ is not very good space for F.T.

later show $f \in C_c^\infty$ then $\hat{f}$ cannot be compactly supp.

Definition. (Schwartz)

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \begin{array}{l} f : \mathbb{R}^n \to \mathbb{C} \text{ smooth} \\ \sup\limits_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha f| \le C_{\alpha,\beta} \end{array} \right\}$$

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}. \quad \beta = (\beta_1, \cdots, \beta_n)$$

Example

1) $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$

2) $f(x) = e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$

Theorem: 1) $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$ $\forall \alpha$.

$$\widehat{\partial_j f}(\xi) = i\xi_j \hat{f}(\xi)$$

proof:

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = \int e^{-ix\cdot\xi} \frac{\partial f}{\partial x_j}(x)\,dx$$

$$\overset{\text{integrate by parts}}{=} \int i\xi_j e^{-ix\cdot\xi} f(x)\,dx = i\xi_j \hat{f}(\xi) \quad \square$$

no boundary term since f decays fast

In particular, $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$

$$\Rightarrow \widehat{\Delta f}(\xi) = \sum_{i=1}^{n} \widehat{\frac{\partial^2 f}{\partial x_i^2}}(\xi) = -\sum_j \xi_j^2 \hat{f}(\xi)$$
$$= -|\xi|^2 \hat{f}(\xi)$$

2) $\frac{\partial}{\partial \xi_j} \hat{f}(\xi) = -i\, \widehat{x_j f}(\xi)$

proof: $\partial_j \hat{f}(\xi) = \partial_j \int f(x) e^{-ix\cdot\xi} dx$

$$\overset{\text{LDCT}}{=} \int f(x)(-ix_j e^{-ix\cdot\xi})dx$$

$$= -i\, \widehat{x_j f}(\xi) \quad \square$$

Example: F.T. of Gaussian $e^{-|x|^2/2}$:

1D: $f(x) = e^{-x^2/2}$. Note $\frac{d}{dx} f + xf = 0$.

Take F.T. on both sides:

$$i\xi \hat{f}(\xi) + i \frac{d}{d\xi} \hat{f}(\xi) = 0.$$

$\Rightarrow \hat{f}(\xi)$ satisfies the same ODE as f.

$\Rightarrow \hat{f} = cf$

$$\hat{f}(0) = cf(0) = c = \int e^{-x^2/2} dx = \sqrt{2\pi}$$

$$\Rightarrow \hat{f} = \sqrt{2\pi}\, e^{-\xi^2/2}$$

n D: $$\hat{f}(\xi) = \int e^{-(x_1\xi_1 + \dots + x_n\xi_n)} e^{-x_1^2/2} e^{-x_2^2/2} \cdots e^{-x_n^2/2} dx$$

$$= \prod_{i=1}^{n} \int e^{-\xi_i x_i} e^{-x_i^2/2} dx_i$$

$$= \prod_{i=1}^{n} \sqrt{2\pi}\, e^{-\xi_i^2/2}$$

$$= (2\pi)^{\frac{n}{2}} e^{-|\xi|^2/2}$$

3) $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{f} \in \mathcal{S}(\mathbb{R}^n)$

proof: $$\xi^\beta \partial^\alpha \hat{f}(\xi) = \xi^\beta (-i)^{|\alpha|} \widehat{x^\alpha f}(\xi)$$

$$= (-i)^{|\alpha|} (i)^{|\beta|} \widehat{\partial^\beta (x^\alpha f)}(\xi)$$

$$\sup_{\xi \in \mathbb{R}^n} |\xi^\beta \partial^\alpha \hat{f}(\xi)| \le \sup_{\xi \in \mathbb{R}^n} |\widehat{\partial^\beta (x^\alpha f)}(\xi)|$$

$$f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \partial^\beta (x^\alpha f) \in \mathcal{S}(\mathbb{R}^n)$$

It suffices to show $g \in \mathcal{S}(\mathbb{R}^n)$

$$\sup_{\xi \in \mathbb{R}^n} |\hat{g}| \le +\infty$$

$$|\hat{g}| = \left|\int e^{-ix\cdot\xi} g(x)\,dx\right| \le \|g\|_\infty < +\infty$$

since $g \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Hence $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear operator over Schwartz space

proposition. F.T. under linear trans.

Take $A \in GL(n, \mathbb{R})$ $f \in \mathcal{S}(\mathbb{R}^n)$

$$f_A(x) := f(Ax) \in \mathcal{S}(\mathbb{R}^n)$$

Then

$$\hat{f}_A(\xi) = \frac{1}{|\det(A)|} \hat{f}\left((A^T)^{-1}\xi\right)$$

proof:

$$\hat{f}_A(\xi) = \int e^{-ix\cdot\xi} f_A(x)\,dx$$

$$= \int e^{-ix\cdot\xi} f(Ax)\,dx \qquad y = Ax,\quad dy = |\det A|\,dx$$

$$= \int \frac{1}{|\det A|} \cdot e^{-i(A^{-1}y)\cdot\xi} f(y)\,dy$$

$$= \frac{1}{|\det A|} \int e^{-iy\cdot\left((A^{-1})^T\xi\right)} f(y)\,dy$$

$$= \frac{1}{|\det A|} \hat{f}\left((A^{-1})^T\xi\right)$$

Examples 1) $Ax = ax$, $a \neq 0$. $x \in \mathbb{R}^n$

$$\hat{f}_A(\xi) = \frac{1}{|a|^n} \hat{f}\left(\frac{\xi}{a}\right)$$

2) $\widehat{e^{-\frac{\varepsilon|x|^2}{2}}} = ?$

$f(x) = e^{-\frac{|x|^2}{2}}$, $a = \sqrt{\varepsilon}$ as in (1)

$$\Rightarrow \widehat{e^{-\frac{\varepsilon|x|^2}{2}}} \underset{\text{verified later}}{=} \frac{1}{\varepsilon^{n/2}} \frac{1}{(2\pi)^{n/2}} e^{-\frac{|\xi|^2}{2\varepsilon}}$$

3) $O \in SO(n)$. $O^{-1} = O^T$. $\det O = 1$

$$\hat{f}_O(\xi) = \hat{f}((O^{-1})^T\xi) = \hat{f}(O\xi)$$

Hence F.T. is invariant (commute with) under rotations

proposition Translation $a \in \mathbb{R}^n$. $f \in \mathcal{S}(\mathbb{R}^n)$ $f_a(x) := f(x+a)$

$$\Rightarrow \hat{f}_a(\xi) = e^{ia\xi} \hat{f}(\xi)$$

proof is omitted.

* Proposition. $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$\int \hat{f}(\xi)\, g(\xi)\, d\xi = \int f(\xi)\, \hat{g}(\xi)\, d\xi$$

proof: Fubini Theorem.

proposition. $f, g \in \mathcal{S}(\mathbb{R}^n)$. then

$$\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$$

proof: Fubini Theorem.

Theorem. (Fourier Inversion on $\mathcal{S}(\mathbb{R}^n)$)

$$f(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi}\, \hat{f}(\xi)\, d\xi.$$

$$\forall f \in \mathcal{S}(\mathbb{R}^n)$$

Corollary. $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear isomorphism.

proof. of Corollary: $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \hat{f}(\mathbb{R}^n)$ ✓

Now if $\hat{f}(\xi) \in \mathcal{S}(\mathbb{R}^n)$.

Inversion formula $\Rightarrow$ $f(x) = \frac{1}{(2\pi)^n} \hat{\hat{f}}(-x)$

Hence $f \in \mathcal{S}(\mathbb{R}^n)$. $\square$

proof of the Theorem:

$$\text{RHS} = \frac{1}{(2\pi)^n} \int e^{ix\xi} \hat{f}(\xi)\, d\xi$$

$e^{ix\xi} \notin \mathcal{S}(\mathbb{R}^n)$. cannot apply $\int \hat{f}(\xi) g = \int \hat{g} \cdot f$

$$= \lim_{\varepsilon \to 0^+} \int e^{-\frac{\varepsilon|\xi|^2}{2}} e^{ix\xi} \hat{f}(\xi)\, d\xi.$$

$$= \lim_{\varepsilon \to 0^+} \int \underbrace{e^{-\frac{\varepsilon|\xi|^2}{2}}}_{\in \mathcal{S}(\mathbb{R}^n)} \underbrace{\hat{f}_x(\xi)}_{\in \mathcal{S}(\mathbb{R}^n)}\, d\xi$$

where $f_x(y) = f(x+y)$.

$$= \lim_{\varepsilon \to 0^+} \int \widehat{e^{-\frac{\varepsilon|\xi|^2}{2}}}(y)\, f_x(y)\, dy$$

$$= \lim_{\varepsilon \to 0^+} (2\pi)^{n/2} \int \boxed{e^{-\frac{|y|^2}{2\varepsilon}} \cdot \frac{1}{\varepsilon^{n/2}}} f(x+y)\, dy$$

δ-like sequence $\to C\delta$ for some const. C

$$= \cdot f(x).$$

Then calculate the const. $C = (2\pi)^{-n/2}$ $\square$

Now suppose $f: \mathbb{R}^n \to \mathbb{C}$

$f = \mathrm{Re} f + i\, \mathrm{Im}(f)$

proposition. $\widehat{\bar{f}}(\xi) = \overline{\hat{f}}(-\xi) \quad f \in \mathcal{S}(\mathbb{R}^n)$.

proof:
$$\widehat{\bar{f}}(\xi) = \overline{\overline{\int e^{-ix\xi} \bar{f}(x)\,dx}}$$
$$= \overline{\int e^{ix\xi} f(x)\,dx}$$
$$= \overline{\hat{f}}(-\xi) \quad \square$$

Definition. $L^2(\mathbb{R}^n) := \{ f: \mathbb{R}^n \to \mathbb{C} \mid \int |f|^2 dx < +\infty \}$.

$$\|f\|_{L^2} = \left(\int |f|^2 dx \right)^{\frac{1}{2}}$$

$$\langle f, g \rangle = \int f(x) \bar{g}(x)\,dx.$$

Note:
$$\langle \hat{f}, \hat{g} \rangle = \int \hat{f}(\xi)\, \overline{\hat{g}}(\xi)\,d\xi$$
$$= \int \hat{f}\ \widehat{\bar{g}}(-\xi)\,d\xi$$
$$= \int \hat{f}(x)\, \bar{g}(-x)\,dx$$

Recall $\int \hat{f} g = \int \hat{g} \cdot f$

$f, g \in \mathcal{S}(\mathbb{R}^n)$

Inversion formula $\overset{}{=} (2\pi)^n \int f(-x)\, \bar{g}(-x)\, dx$

$$= \int f(x)\, \bar{g}(x)\, dx = (2\pi)^n \langle f, g \rangle$$

$\Rightarrow$ Fourier Transform preserves L^2-inner product on $\mathcal{S}(\mathbb{R}^n)$.

(Plancherel)

Corollary. $\|f\|_{L^2} = (2\pi)^{-n} \|\hat{f}\|_{L^2}$.

Exercise: $\mathcal{S}(\mathbb{R}^n)$ is dense in L^2.

Definition. F.T. on L^2 space

Take $f \in L^2(\mathbb{R}^n)$ Let $\{\varphi_n\} \in \mathcal{S}(\mathbb{R}^n)$

$\varphi_n \to f$ in L^2 then $\hat{\varphi}_n$ is Cauchy in L^2 by Plancherel

Define

$$\hat{f} := \lim_{n\to\infty} \hat{\varphi}_n \quad \text{in } L^2\text{-sense},$$

Exercise: Check independence of $\{\varphi_n\}$.

Proposition. $\widehat{fg}(\xi) = (2\pi)^{-n}\, \hat{f} * \hat{g}(\xi)$

proof: Recall $\widehat{f * g}(\xi) = \hat{f}\ \hat{g}$

take F.T. on LHS:

$$\hat{\hat{fg}} = (2\pi)^{n} f(-x)\, g(-x)$$

take F.T. on RHS:

$$(2\pi)^{-n}\, \widehat{\hat{f} * \hat{g}} = \hat{\hat{f}}(x) \cdot \hat{\hat{g}}(x) \cdot (2\pi)^{-n}$$

$$= (2\pi)^{\cancel{2}n} f(-x)\, g(-x) \cdot \cancel{(2\pi)^{-n}}$$

Hence F.T. of LHS = F.T. of RHS.

$\Longleftrightarrow$ LHS = RHS, by bijectivity of F.T.

$\square$

Summary F.T. on $\mathcal{S}(\mathbb{R}^n)$

a) $\widehat{\partial_j f}(\xi) = i\xi_j \hat{f}(\xi)$

b) $\frac{\partial}{\partial \xi_j}\hat{f}(\xi) = -i\,\widehat{x_j f}(\xi)$

c) $A \in GL(n,\mathbb{R})$. $f_A(x) = f(Ax)$

$\hat{f_A}(\xi) = \frac{1}{|\det A|}\hat{f}((A^{-1})^T\xi)$

d) $a \in \mathbb{R}^n$. $f_a(x) = f(a+x)$

$\hat{f_a}(\xi) = e^{ia\xi}\hat{f}(\xi)$

e) $\int \hat{f} g = \int \hat{g} f$.

f) $f(x) = \int \hat{f}(\xi) e^{ix\cdot\xi} d\xi \cdot \frac{1}{(2\pi)^n}$

g) $\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x)$

h) $\hat{\bar{f}}(\xi) = \bar{\hat{f}}(-\xi)$

i) $\widehat{f * g} = \hat{f}\,\hat{g}$

j) $\widehat{fg}(\xi) = (2\pi)^{-n}\,\hat{f} * \hat{g}(\xi)$

(k) $\langle \hat{f}, \hat{g} \rangle_{L^2} = (2\pi)^n \langle f, g \rangle_{L^2}$

Definition (Topology of Scwartz Space)

$\{\varphi_n\}_{n=1}^{\infty} \in \mathcal{S}(\mathbb{R}^n)$

$\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ if

$\forall \alpha, \beta.$ $\|\varphi_n\|_{\alpha,\beta} \to \|\varphi\|_{\alpha,\beta}$

i.e. $\sup_{x\in\mathbb{R}^n} \| x^\beta \partial^\alpha \varphi_n| \to \sup_{x\in\mathbb{R}^n} | x^\beta \partial^\alpha \varphi|$

proposition: F.T. is continuous on $\mathcal{S}(\mathbb{R}^n)$.

proof: take $f_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$

Claim: $\hat{f}_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$.

$$\xi^\beta \partial^\alpha \hat{f}_j(\xi) \approx \xi^\beta (-i)^{|\alpha|} \widehat{x^\alpha f_j}(\xi)$$

$$= (-i)^{|\alpha|+|\beta|} \widehat{\partial^\beta (x^\alpha f_j)}(\xi)$$

$$|\xi^\beta \partial^\alpha \hat{f}_j(\xi)| = |\widehat{\partial^\beta |x^\alpha f_j|}| = \left| \int e^{-ix\xi} \partial^\beta (x^\alpha f_j) dx \right|$$

$$\leq \int |\partial^\beta (x^\alpha f_j)| \, dx$$

$$= \int |\partial^\beta (x^\alpha f_j)| \cdot (1+|x|^2)^{-N} \cdot (1+|x|^2)^N \, dx$$

$$\lesssim \underbrace{\sup |\partial^\beta (x^\alpha f_j) \cdot (1+|x|^2)^N|}_{\to 0 \text{ as } f_j \to 0} \int \left(\frac{1}{1+|x|^2}\right)^N dx$$

Choose large N s.t. the above integral $< +\infty$.

Done. $\square$

RMK. The last theorem says F.T. is a linear homeomorphism on $\mathcal{S}(\mathbb{R}^n)$. $(\mathcal{F}^{-1} f)(\xi) = c\, \hat{f}(-\xi)$ (c: const.)

This allows us to generalize F.T. to (tempered) distributions via duality.

Now we are going to generalize F.T. on (tempered) distributions.

Definition. Tempered distribution $\mathcal{S}'(\mathbb{R}^n)$

$\{T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ linear, continuous$\}$

$\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$
$\Rightarrow T(\varphi_n) \to T(\varphi)$ in $\mathbb{C}$.

Examples: 1) $T \equiv 1$: $T(\varphi) = \int \varphi \, dx$

2) $\mathcal{E}'(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$.

$u \in \mathcal{E}'(\mathbb{R}^n) \Rightarrow u(f)$ is well-defined
$= (C^\infty)'$

3) $f \in L^1_{loc}(\mathbb{R}^n)$. $|f(x)| \le C|x|^M$, for large x.
$(M > 0)$

$\Rightarrow f \in \mathcal{S}'(\mathbb{R}^n)$

$$\int f\varphi = \underbrace{\int_{|x|\le N} f\varphi}_{<+\infty} + \underbrace{\int_{|x|\ge N} f\varphi}_{|\ | < \int |x|^M \varphi < +\infty} < +\infty \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

4) Non-example: $f = e^x$.

Definition. F.T. on tempered distributions

$$\hat{T}(\varphi) := T(\hat{\varphi}), \quad T \in \mathcal{S}'(\mathbb{R}^n)$$

Motivation: $\int \hat{f} g = \int f \hat{g}$. $\quad \varphi \in \mathcal{S}(\mathbb{R}^n)$

Examples

1) $\hat{\delta}(\varphi) = \delta(\hat{\varphi}) = \hat{\varphi}(0) = \int \varphi(x)\,dx$

$\Rightarrow \hat{\delta} \equiv 1$ in $\mathcal{S}'(\mathbb{R}^n)$

2) $\hat{1}(\varphi) = 1(\hat{\varphi}) = \int \hat{\varphi}(\xi)\,d\xi \overset{\text{Inversion formula}}{=} (2\pi)^n f(0)$.

$\Rightarrow \hat{1} = (2\pi)^n \delta$ in $\mathcal{S}'(\mathbb{R}^n)$.

RMK. $\hat{\hat{\delta}} = (2\pi)^n \delta$.

Proposition. $u \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow \hat{u} \in \mathcal{S}'(\mathbb{R}^n)$

proof: take $f \in \mathcal{S}(\mathbb{R}^n)$

$$\hat{u}(f) = u(\hat{f}).$$

Let $f_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, then $\hat{f}_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$

$$\hat{u}(f_j) = u(\hat{f}_j) \to u(0) = 0$$

since u is continuous on $\mathcal{S}(\mathbb{R}^n)$.

proposition. $\widehat{\frac{\partial}{\partial x_j} u} = i \xi_j \hat{u}$

proof:

$$\widehat{\frac{\partial}{\partial x_j} u}(f) = \left(\frac{\partial}{\partial x_j} u\right)(\hat{f}) = -u\left(\frac{\partial}{\partial x_j} \hat{f}\right)$$

$$= -u\left(-i \widehat{\xi_j f}\right)$$

$$= i u\left(\widehat{\xi_j f}\right)$$

$$= i \xi_j \hat{u}(f)$$

$\square$

proposition. $\frac{\partial}{\partial \xi_j} \hat{u} = -i \widehat{x_j u}$

proof: $\left(\frac{\partial}{\partial \xi_j} \hat{u}\right)(f) = -\hat{u}\left(\frac{\partial}{\partial x_j} f\right)$

$$= -u\left(\widehat{\partial_j f}\right)$$

$$= -u(i x_j \hat{f})$$

$$= -i\, x_j u(\hat{f}).$$

$$= -i\, \widehat{x_j u}(f) \quad \square$$

$\ast$ **Theorem.** F.T. $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a linear homeomorphism

with $\left(\mathcal{F}^{-1}(u)\right)(\varphi) = \frac{1}{(2\pi)^n}\, \hat{u}(\tilde{\varphi})$.

The same formula as functions

proof. Claim: $\hat{\hat{f}} = f$. just inversion formula on $\mathcal{S}(\mathbb{R}^n)$

Then the Theorem follows.

Now if $u \in \mathcal{S}'(\mathbb{R}^n)$, $f \in C^\infty(\mathbb{R}^n)$ in general $f \cdot u \notin \mathcal{S}'(\mathbb{R}^n)$.

$(fu)(g) = u(fg)$. fg may not be in Schwartz.

Definition. $f \in C^\infty(\mathbb{R}^n)$, $|f| \le C x^N$.

then if $g \in \mathcal{S}(\mathbb{R}^n)$, $fg \in \mathcal{S}(\mathbb{R}^n)$ as well.

Define $(f \cdot u)(g) := u(fg)$