EGOROV, LUSIN and HAUSDORFF.

Topic 1. Lusin Thm.

Let μ be a Borel regular measure in $\mathbb{R}^n$. $f: \mathbb{R}^n \to \mathbb{R}^m$ be μ-measurable. A is μ-measurable. $\mu(A) < +\infty$.

Fix $\varepsilon > 0$. Then there exists a compact set $K \subseteq A$. such that.

(i) $\mu(A \setminus K) < \varepsilon$ (ii) $f|_K$ is cts.

Proof. Step 1. Construct a compact subset of A.

Fix $i \in \mathbb{N}$. Let $B_{ij} \subseteq \mathbb{R}^m$ be disjoint subsets s.t. $\bigcup_{j \geq 1} B_{ij} = \mathbb{R}^m$. $\operatorname{diam} B_{ij} < \frac{1}{i}$

Let $A_{ij} = f^{-1}(B_{ij}) \cap A$. Then A_{ij} is μ-measurable. $A = \bigcup_{j=1}^{\infty} A_{ij}$

Write $\nu = \mu|_A$ (μ restrict to A) i.e. $\nu(X) = \mu(X \cap A)$ ν is Radon measure (finite. borel regular).

There exists $K_{ij} \subseteq A_{ij}$ compact. $\nu(A_{ij} \setminus K_{ij}) < \frac{\varepsilon}{2^{i+j}}$

$$\mu\Big(A \setminus \bigcup_{j \geq 1} K_{ij}\Big) = \nu\Big(A \setminus \bigcup_{j \geq 1} K_{ij}\Big) = \nu\Big(\bigcup_{j \geq 1} A_{ij} \setminus \bigcup_{j \geq 1} K_{ij}\Big) \leq \nu\Big(\bigcup_{j \geq 1} (A_{ij} \setminus K_{ij})\Big) \leq \sum_{j \geq 1} \frac{\varepsilon}{2^{i+j}} = \frac{\varepsilon}{2^i}$$

$$\bigcup_{j \geq 1} A_{ij} - \bigcup_{j \geq 1} K_{ij} \subseteq \bigcup_{j \geq 1} (A_{ij} \setminus K_{ij})$$

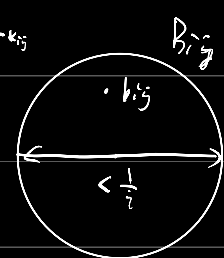
Warning. $\bigcup_{j \geq 1} K_{ij}$ may not be compact. Since $\lim_{N \to \infty} \mu\Big(A \setminus \bigcup_{j=1}^{N} K_{ij}\Big) = \mu\Big(A \setminus \bigcup_{j \geq 1} K_{ij}\Big) < \frac{\varepsilon}{2^i}$. there exists $N_i \in \mathbb{N}$ s.t.

$$\mu\Big(A \setminus \bigcup_{j=1}^{N_i} K_{ij}\Big) < \frac{\varepsilon}{2^{i-1}}$$ Now finite union of compact set is compact. Call $D_i := \bigcup_{j=1}^{N_i} K_{ij}$, compact.

Step 2. construct a cts. function.

Fix $b_{ij} \in B_{ij} \subseteq \mathbb{R}^m$. $\operatorname{diam}(B_{ij}) < \frac{1}{i}$ Define $g_i : D_i \to \mathbb{R}^m$. $g_i(x) = b_{ij}$ for $x \in K_{ij}$. $1 \leq j \leq N(i)$

$$= \sum_{j=1}^{N(i)} b_{ij} \mathbb{1}_{K_{ij}}$$



$|f(x) - g_i(x)| < \frac{1}{i}$ since if $x \in D_i$, $f(x) \in B_{ij}$ for some j.

Let $K := \bigcap_{i=1}^{\infty} D_i$ still compact. $\mu(A \setminus K) = \mu\Big(A \setminus \bigcap_{i=1}^{\infty} D_i\Big) \leq \sum_{i=1}^{\infty} \mu(A \setminus D_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i-1}} < 2\varepsilon$. (无伤大雅)

$g_i \rightrightarrows f$ uniformly on K. $g := f|_K$ is cts.

Topic 2. Egorov Thm.

$f_k : X \to \mathbb{R}^n$ be μ-measurable functions. $f_k \to f$ a.e. on X. $\mu(X) < +\infty$ Then there exists $B \subseteq A$ s.t.

$\forall \varepsilon.\ \exists N\ \forall n \geq N \Updownarrow$ $\quad |f(x) - f_n(x)| < \varepsilon.$

(i). $\mu(A \setminus B) < \varepsilon$. (ii) $f_k \rightrightarrows f$ on B. $\quad \mu\left(\bigcap_{i\geq 1} \bigcup_{j \geq 1} \bigcap_{k \geq j} \{x \in X \mid |f_k(x) - f(x)| < \frac{1}{i}\}\right) = \mu(X).$

Proof. $C_{ij} = \bigcup_{k=j}^{\infty} \{x \in X : |f_k(x) - f(x)| > \frac{1}{2^i}\}$, i, j fixed.

key: $\mu(X) < +\infty$

So $C_{i,j+1} \subseteq C_{i,j}$ decreasing set. So $\lim_{j \to \infty} \mu(C_{ij}) = \mu\left(\bigcap_{j=1}^{\infty} C_{ij}\right) = 0$ by pointwise conv.

$\Rightarrow \exists N(i) \in \mathbb{N}$ s.t. $\mu(C_{i,N(i)}) < \frac{\varepsilon}{2^i}$ Let $B^c := \bigcup_{i \geq 1} C_{i,N(i)}$. $\quad \mu(B^c) \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$

$\forall x \in B$: $x \in \bigcap_{i \geq 1} C_{i,N(i)}$, So $|f(x) - f_n(x)| < \frac{1}{2^i}$ if $n > N(i) \Leftrightarrow$ unif. conv

Alternative similar proof: $f_n \xrightarrow{a.e.} f \Leftrightarrow \mu\left[\left(\bigcap_{k \geq 1} \bigcup_{N \geq 1} \bigcap_{n \geq N} \{x : |f(x) - f_n(x)| < \frac{1}{k}\}\right)^c\right] = 0$

$$\Leftrightarrow \mu\left(\bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{n \geq N} \{|f_n(x) - f(x)| \geq \tfrac{1}{k}\}\right) = 0.$$

$$\Longrightarrow \mu\Big(\underbrace{\bigcap_{N \geq 1} \bigcup_{n \geq N} \{|f_n(x) - f(x)| \geq \tfrac{1}{k}\}}_{E_N \text{ decreasing set sequence}}\Big) = 0 \quad \forall k.$$

$\mu(X) < +\infty$

$$\lim_{N \to \infty} \mu\left(\bigcup_{n \geq N} \{|f_n(x) - f(x)| \geq \tfrac{1}{k}\}\right) = 0$$

$\Leftarrow \exists N_k,\ \mu\left(\bigcup_{n \geq N_k} |f_n(x) - f(x)| \geq \frac{1}{k}\}\right) < \frac{\varepsilon}{2^{k+1}}$ $\quad \mu(E_{N_k}) < \frac{\varepsilon}{2^{k+1}}$. $\quad E := \bigcup_{k \geq 1} E_{N_k}$. $\Rightarrow \mu(E) \leq \sum_{k \geq 1} \mu(E_{N_k}) = \frac{\varepsilon}{2} < \varepsilon$

Claim: $f_n \rightrightarrows f$ on $X \setminus E$.

Topic 3 Hausdorff measure.

(i) Let $A \subseteq \mathbb{R}^n$. $\quad 0 \leq s < +\infty \quad 0 < \delta \leq +\infty$.

$$H^s_\delta(A) = \inf\left\{\sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \,\middle|\, A \subseteq \bigcup_{j=1}^{\infty} C_j.\ \operatorname{diam} C_j < \delta\right\} \quad \text{Here } \alpha(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$$

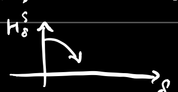
(ii) The Hausdorff measure of A

$$H^s(A) := \sup_{\delta > 0} H^s_\delta(A) = \lim_{\delta \to 0} H^s_\delta(A).$$

RMK. 1) $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx \quad (0 < s < \infty)$ Gamma function.

2) $\operatorname{vol}(B(x,r)) = \alpha(n) r^n$. where $B(x,r)$ is the ball in $\mathbb{R}^n$ with radius r.

3) $H^s_\delta \downarrow$ in δ



Theorem. Hausdorff measure H^s is Borel regular. for $0 \le s < +\infty$

Proof. (i) Borel 集都可测. See Evans. 3043 Homework 2.

(ii) Borel regularity: Given A. $\exists$ Borel set $B \subseteq A$ $\quad H^s(B) = H^s(A)$.

$$H^s_\delta(A) = \inf\left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \,\middle|\, A \subseteq \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j < \delta \right\}$$

$$= \inf\left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} \bar{C}_j}{2}\right)^s \,\middle|\, A \subseteq \bigcup_{j=1}^{\infty} \bar{C}_j,\ \operatorname{diam} \bar{C}_j < \delta \right\} \qquad \bar{C}_j \text{ is the closure of } C_j.$$

$\Rightarrow$ We could require C_j be closed.

$$H^s_\delta(A) = \inf\left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \,\middle|\, A \subseteq \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j < \delta,\ C_j \text{ closed} \right\}$$

Let $A \subseteq \mathbb{R}^n$. $H^s(A) < +\infty \Rightarrow H^s_\delta(A) < +\infty$.

Choose closed set $\{C_{kj}\}_{j=1}^{\infty}$ $\operatorname{diam} C_{kj} < \frac{1}{k}$ covering A. and

$$\sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_{kj}}{2}\right)^s < H^s_{\frac{1}{k}}(A) + \frac{1}{k}.$$

δ $\quad$ ε. $\quad \sum \text{dian} \cdots \le H^s_\delta + \varepsilon$.

Let $A_k = \bigcup_{j\ge1} C_{kj}$ $\quad B = \bigcap_{k\ge1} A_k$ $\quad$ Claim: $A \subseteq B$. $\underline{\text{Borel}}$ $H^s(A) = H^s(B)$

trivial

Since $B = \bigcap A_k$, $B \subseteq A_k = \bigcup_{j\ge1} C_{kj}$ $\quad$ for fix k.

$$\Rightarrow H^s_{\frac{1}{k}}(B) \le \sum_{j\ge1} \alpha(s) \left(\frac{\operatorname{diam}(C_{kj})}{2}\right)^s \le H^s_{\frac{1}{k}}(A) + \frac{1}{k}.$$

$$\left.\begin{array}{l} \text{let } k \to +\infty \quad H^s(B) \le H^s(A) \\ \text{Since } A \subseteq A_k\ \forall k, \Rightarrow A \subseteq B.\ \ H^s(B) \ge H^s(A) \end{array}\right\} \text{equality holds}$$

Thm. properties of H^s.

(i) $H^0 \equiv$ counting measure on $\mathbb{R}^n$.

(ii) $H^s \equiv 0$ if $s > n$.

(iii) $H^1 = \lambda^1$ 1D Lebesgue on $\mathbb{R}$. ($H^n = \lambda^n$ on $\mathbb{R}^n$. but the proof is not easy)

(iv) $H^s(\alpha A) = \alpha^s H^s(A)$ for $\alpha > 0$. $A \subseteq \mathbb{R}^n$

(v) $H^s(L(A)) = H^s(A)$ where $A \subseteq \mathbb{R}^n$, $L: \mathbb{R}^n \to \mathbb{R}^n$ is an affine isometry.

Proof: Evans. Chapter 2.

Other details: See Evans.