Elliptic Eqs.

$$\begin{cases} Lu = f & \text{in } U. \\ u = 0 & \text{on } \partial U \end{cases}$$

U is open bounded. f is given $\quad Lu = -\sum \left(a^{ij}(x)\, u_{x_i}\right)_{x_j} + \sum b^i(x)\, u_{x_i} + c(x)\, u \quad$ (Divergence form)

OR

$$Lu = -\sum a^{ij}(x)\, u_{x_i} u_{x_j} + \sum b^i(x)\, u_i(x) + c(x)\, u. \quad (\text{Non-div. form})$$

Assume the symmetry condition: $a^{ij} = a^{ji}$.

Def. We say L is (uniformly) Elliptic if $\exists\ \theta > 0$. s.t. $\sum a^{ij}(x)\xi_i\xi_j \geq \theta |\xi|^2 \quad \forall \xi \in \mathbb{R}^n$, a.e. $x \in U$.

In particular. $A = \left(a^{ij}(x)\right)_{i,j}$ is positive-defined. eigenvalues $\lambda_i \geq \theta$

RMK. $D^2 u = \sum a^{ij} u_{x_i} u_{x_j}$. $\underbrace{F := -A\, Du}_{\text{physical law.}}$ = diffusion flux. $F \cdot Du \leq 0 \iff$ Elliptic condition

Weak solution.

Assume $a^{ij}, b^i, c \in L^\infty(U) \quad \forall\ 1 \leq i,j \leq n \quad ; f \in L^2(U)$

$$Lu = f \qquad Lu = -\sum \left(a^{ij} \cdot u_{x_i}\right)_{x_j} + b^i u_{x_i} + c\,u$$

$\downarrow$ multiply both sides by a test function $v \in C_c^\infty(U)$ and integrate by parts

Motivation: (Assume u is smooth)

$$\int_U \sum a^{ij} u_{x_i} v_{x_j} + \sum b^i u_{x_i} \cdot v + c\,u\,v \, dx = \int_U f v \, dx \quad (\text{bilinear formulation})$$

Definition. The bilinear form $B[\ ,\ ]$ ($V \times V \to \mathbb{R}$.) associated with the divergence form elliptic operator L is

$$B[u,v] = \int_U \sum a^{ij} u_{x_i} v_{x_j} + \sum b^i u_{x_i} \cdot v + c\,u\,v \, dx. \quad \forall\ u, v \in H_0^1(U)$$

We say $u \in H_0^1(U)$ is a weak solution of $\begin{cases} Lu = f & \text{on } U \\ u = 0 & \text{on } \partial U. \end{cases}$ if $B[u,v] = (f, v)_{L^2(U)} \quad \forall v \in H_0^1(U)$

$$= \int_U f v \, dx$$

RMK. Sometimes this set-up is called the $\underbrace{\text{variational formulation}}_{\text{minimizer of some energy functional}}$ of the Boundary value problem.

More generally, one may consider the problem

$$\begin{cases} Lu = \underbrace{f^0 - \sum_{i=1}^{n} f^i_{x_i}}_{\in H^{-1} \text{ if } f^i \in L^2} & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Def. We say $u \in H_0^1(U)$ is a weak solution to the BVP if

$B[u,v] = \langle \underset{\in H^{-1}(U)}{f}, \underset{\in H_0^1(U)}{v} \rangle \quad \forall\, v \in H_0^1(U)$ we identify $f \in H^{-1}(U)$, the dual of $H_0^1(U)$

$$= \int_U f^0 v + \sum_{i=1}^{n} f^i v_{x_i}\, dx$$

RMK. Other boundary value conditions can be transformed into the simple setting as above.

Suppose ∂U is C^1. $u \in H^1(U)$ is a weak solution of $\begin{cases} Lu = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$ in the trace sense.

Take g to be the trace of $w \in H^1(U)$

$\tilde{u} := u - w$. Then $\begin{cases} L\tilde{u} = \tilde{f} = f - Lw & \text{in } U. \\ \tilde{u} = 0 & \text{on } \partial U \end{cases}$

$w \in H^1 \Rightarrow Lw \in H^{-1}$. So it really needs to discuss H^{-1} space if one wants to solve general BVP.

Existence of weak solutions Lax-Milgram.

Assume H: $\mathbb{R}$-Hilbert Space. norm $\|\cdot\|$, inner product $(\ ,\ )$

Theorem (Riesz Representation) f is a bounded linear function on H. Then $\exists!\, u \in H$ s.t. $f(v) = (u, v) \quad \forall\, v \in H$.

RMK. The result is obvious if H is separable with a basis $(e_n)_{n\in\mathbb{N}}$. $f(v) = f(\sum v^i e_i) = \sum v^i f(e_i)$

So take $u = \sum_{i=1}^{\infty} f(e_i) e_i$. conv. in L^2-sense

- If H is NOT separable, assume u exists, and then work backwards. $H = \langle u \rangle \oplus u^\perp$

Theorem (Lax-Milgram, essential part) Suppose $B: H\times H \to \mathbb{R}$ is a bilinear form.
Assume $|B[u,v]| \le \alpha \|u\| \|v\|$ for some $\alpha>0$. $\forall u,v \in H$. Then

$$B[u,v] = (Au, v) \quad \text{where} \quad A: H\to H \text{ is a bounded linear operator.}$$

Furthermore, if B is coercive i.e. $B[u,u] \ge \delta \|u\|^2$ for some $\delta>0$. Then A is invertible and $\|A^{-1}\| \le \frac{1}{\delta}$.

Proof. 1. For fixed $u\in H$, $B[u,\cdot]$ is a bounded linear functional on H.

$\xRightarrow{\text{Riesz}}$ we have $B[u,\cdot] = l_u(\cdot) = (Au,\cdot)$ $u \mapsto Au$ is hence well-defined.

A is linear because: $\big(A(\lambda_1 u_1 + \lambda_2 u_2), v\big) = B[\lambda_1 u_1 + \lambda_2 u_2, v]$

$$= \lambda_1 B[u_1,v] + \lambda_2 B[u_2,v] = \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v).$$

A is bounded: take $v = Au$. $\|Au\|^2 = B[u, Au] \le \alpha \|u\| \|Au\|$

$$\Rightarrow \|Au\| \le \alpha \|u\| \Rightarrow \|A\| \le \alpha$$

2. If $B[u,u] \ge \delta \|u\|^2$:

- A is injective: Suppose $Ax = 0$: $\|x\|^2 \le \frac{1}{\delta} B[x,x] = \frac{1}{\delta}(Ax,x) \le \frac{1}{\delta}\|Ax\| \|x\| \Rightarrow x = 0$.
- A is surjective:
  - $\mathrm{Ran}(A)$ is closed: if $Ax_n \to y$, $\|x_n - x_m\|$ is Cauchy. Hence $x := \lim_{n\to\infty} x_n \in H$.
    $Ax = y$ by continuity.
  - Surjectivity: Suppose $z \in \mathrm{Ran}(A)^\perp$.
    $\|z\|^2 \le \frac{1}{\delta} B[z,z] = \delta^{-1}(Az, z) = 0 \Rightarrow z = 0$.

$\Rightarrow \mathrm{Ran}(A) = H$, the whole space. $\Rightarrow A^{-1}$ exists. $\|A^{-1}\| \le \delta^{-1}$.

■

RMK. If $B[\ ,\ ]$ is symmetric: $B[u,v] = B[v,u]$. Then $A = A^*$. A is symmetric.
$((u,v))_H$ is a new inner product on H. One can prove Lax-Milgram by Riesz directly.

Corollary. (Lax - Milgram, usual version)

Let f be a bounded linear functional on H with paring $f(v) = \langle f, v \rangle$

If B is bounded, coercive bilinear form on H. Then the equation

$$B[u,v] = \langle f, v \rangle \quad \forall\, v \in H.$$

has a unique solution u in H.

Proof: $B[u,v] = (Au, v)$ by previous. where A bounded and invertible.

$\overset{?}{=} \langle f, v \rangle$

By Riesz. $\langle f, v \rangle = (w_f, v)$ $\quad w_f \in H$ unique.

Hence solve $Au = w_f \Rightarrow u = A^{-1} w_f$. Done.

Energy Estimate of Elliptic PDEs

$$B[u,v] = \int_U \sum_{1\le i,j\le n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} \cdot v + c\,uv\,dx. \qquad \forall\, u,v \in H_0^1(U)$$

Theorem For some consts. $\alpha, \beta > 0$ $\gamma \ge 0$ we have

(i) $|B[u,v]| \le \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$

(ii) $\beta \|u\|^2_{H_0^1(U)} \le B[u,u] + \gamma \|u\|^2_{L^2(U)}$ $\Leftarrow$ looks like coercive condition

$\forall\, u, v \in H_0^1(U)$

Proof: (i) $|B[u,v]| \le \sum \|a^{ij}\|_{L^\infty} \int_U |Du|\,|Dv|\,dx + \sum \|b^i\|_{L^\infty} \int_U |Du| \cdot |v|\,dx + \|c\|_{L^\infty} \int_U |u|\,|v|\,dx$

$\le \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$

(ii) $B[u,u] = \int_U \underbrace{\sum a^{ij} u_{x_i} u_{x_j}}_{①} + \underbrace{\sum b^i u_{x_i} \cdot u}_{②} + \underbrace{c u^2}_{③}\,dx$

$① \ge \theta \int_U |Du|^2\,dx$ by ellipcity condition.

$|②| + |③| \le \sum \|b^i\|_\infty \int_U |Du|\,|u|\,dx + \|c\|_\infty \|u\|^2_{L^2}$

Cauchy Schwartz

$\le \varepsilon \int_U |Du|^2 + \frac{1}{4\varepsilon} \int_U u^2\,dx$

Choose $\varepsilon > 0$ s.t. $\varepsilon \sum \|b^i\|_\infty \le \frac{\theta}{2}$

$\Rightarrow B[u,u] \geq \frac{\theta}{2}\int |Du|^2 - C\int_U u^2 dx$ ($C>0$ is an appropriate const.)

Now recall Poincaré: $\|u\|_{L^2} \leq C_1 \|Du\|_{L^2}$

$\Rightarrow \quad \|u\|_{H_0^1(U)} \sim \|Du\|_{L^2}$

$\Rightarrow B[u,u] + \gamma \|u\|_{L^2}^2 \geq \beta \|u\|_{H_0^1(U)}$

RMK. $B[\cdot] + \gamma\|\cdot\|_{L^2}^2$ is coercive, but $B[\cdot]$ itself may not be coercive in general.

Thm (Existence of weak solutions)

There exists $\gamma \geq 0$ s.t. for each $\mu \geq \gamma$ and each function $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of BVP $\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$

Proof. $B_\mu[u,v] = B[u,v] + \mu(u,v)$ where $u,v \in H_0^1(U)$

Then apply Lax-Millgram.

applicable thanks to energy estimate.

RMK. Similarly $\begin{cases} Lu + \mu u = \underbrace{f^0 - \sum f^i_{x_i}}_{\in H^{-1}(U)} & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$, $f^i \in L^2(U)$

admits a unique weak solution in $H_0^1(U)$.

Abstractly: $(L + \mu \cdot Id)\underset{\in H_0^1(U)}{u} = \underset{\in H^{-1}(U)}{f}$

i.e. $L + \mu Id : H_0^1(U) \to H^{-1}(U)$ is a linear isomorphism

RMK If $Lu = -\Delta u$. $B[u,v] = \int_U Du \cdot Dv\, dx$

$-\Delta : H_0^1(U) \to H^{-1}(U)$ is an isomorphism.

no 1st order term

If $Lu = -\sum (a^{ij} u_{x_i})_{x_j} + cu$ $(c \geq 0)$ $L: H_0^1 \to H^{-1}$ is an iso.

Fredholm Alternative for compact operators (Adjoint perspective)

X, Y = $\mathbb{R}$-Banach Spaces $\quad H = \mathbb{R}$-Hilbert Space with $(\ ,\ )$ inner product

Theorem (Fredholm alternative)

Let $K: H \to H$ be a compact linear operator, i.e.

$\{x_n\}$ bounded in $H \Rightarrow \{Kx_n\}$ contains a conv. subsequence.

Then

(i) $N(I-K) = \{u \mid (I-K)u = 0\}$ is finite dimensional

(ii) $R(I-K)$ is finite dimensional

(iii) $R(I-K) = N(I-K^*)^{\perp}$

(iv) $N(I-K) = \{0\}$ iff $R(I-K) = H$

(v) $\dim N(I-K) = \dim N(I-K^*)$

Def. (i) Adjoint operator L^* is defined as

$$L^* v = -\sum (a^{ij} v_{x_j})_{x_i} - \sum b^i v_{x_i} + (c - \sum b^i_{x_i}) v$$

provided $b^i \in C^1(\bar{U})$ $1 \le i \le n$.

Motivation. $(v \in C_c^\infty)$ $\displaystyle\int_U Lu \cdot v \, dx = \int_U -\left(\sum a^{ij} u_{x_i}\right)_{x_j} v + \sum b^i u_{x_i} v + cuv \, dx$

(Note: $u|_{\partial U} = 0$. no boundary terms)

$$\overset{\text{integration by parts}}{=} \int_U u \cdot L^* v \, dx$$

(ii) adjoint bilinear form $B^*: H_0^1 \times H_0^1 \to \mathbb{R}$ is defined as

$$B^*[u, v] := B[v, u], \quad \forall u, v \in H_0^1(U)$$

(iii) We say $v \in H_0^1(U)$ is a weak sol. to the adjoint BVP $\begin{cases} L^* v = f & \text{in } U \\ v = 0 & \text{in } \partial U \end{cases}$ if $\underbrace{B^*[v, u]}_{=B[u,v]} = (f, u)$ $\forall u \in H_0^1(U)$

Thm (General Existence Thm of weak sol.) Fredholm dichotomy of Fred. Alternative

(i) One of the following holds:

either ① For each $f \in L^2(U)$, $\exists!$ $u \in H_0^1(U)$ to $\begin{cases} Lu = f & \text{in } U. \\ u = 0 & \text{on } \partial U \end{cases}$

OR ② $\exists$ weak sol. u ($u \not\equiv 0$) of $\begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$

(ii) If $\exists$ non-trivial sol. $\begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ Then $\dim(N(L)) = \dim(N(L^*)) < \infty$.

(iii) $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ has a weak sol. iff $(f, v) = 0$ $\forall v \in N(L^*)$.

Proof of (i)

Chose $\mu = \gamma$ as before. Consider $L_\gamma u := Lu + \gamma u$. Then $\begin{cases} L_\gamma u = g & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ is solvable $\forall g \in L^2(U)$.

Now assume $u \in H_0^1(U)$ is a weak sol. of $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$ $\Longleftrightarrow$ $\begin{cases} Lu + \gamma u = \underbrace{f + \gamma u}_{g} & \text{in } U. \\ u = 0 & \text{on } \partial U \end{cases}$

$u = L_\gamma^{-1}(\gamma u + f)$

$u - \underbrace{\gamma L_\gamma^{-1}}_{K} u = \underbrace{L_\gamma^{-1} f}_{h}$

$u - Ku = h.$

Claim: $K : L^2 \to L^2$ is bounded. linear. compact operator.

Proof: $\beta \|u\|^2_{H_0^1(U)} \le B_\gamma[u,u] = (g, u) = \|g\|_{L^2} \|u\|_{L^2} \le \|g\|_{L^2} \|u\|_{H_0^1}$

$\Rightarrow \|Kg\|_{H_0^1} \lesssim \|g\|_{L^2}$

Since $H_0^1(U) \overset{c}{\hookrightarrow} L^2(U)$ by Rellish-Kontradov, K is compact operator.

Now apply Fred. Alternative (iv) to $(I - K)u = h$.

either (a) $\begin{cases} u - Ku = h \text{ has} \\ \text{a unique solution } u \in L^2(U) \quad \forall h \in L^2(U) \end{cases}$

or (b) $\begin{cases} u - ku = 0 \\ \text{has a non-zero solution in } L^2(U). \end{cases}$

Translate back to Lu and f. Done. ∎

Proof of (ii):

By Fredholm alternative to the operator $I-K$, either $\begin{cases} u-ku=h \text{ has a} \\ \text{unique sol. } u\in L^2(U) \ \forall h \end{cases}$ or $\begin{cases} u-ku=0 \text{ has} \\ \text{a non-trivial sol.} \\ \text{in } L^2(U) \end{cases}$

If $u-ku=0$ has a nontrivial sol. in $L^2(U)$

$\Longleftrightarrow \begin{cases} Lu=0 \text{ in } U \\ u=0 \text{ on } \partial U \end{cases}$ has a non-zero weak sol. in $H_0^1(U)$

From general Fredholm theory,

$$\dim\{u\in L^2(U) \mid u-ku=0\} = \dim\{v\in L^2 \mid v-k^*v=0\} < +\infty$$

Proof of (iii)

If $\begin{cases} Lu=f & \text{on } U \\ u=0 & \text{on } \partial U \end{cases}$ has a weak sol., then

$$\underset{\displaystyle (u,\, L^*\varphi)}{(Lu,\varphi)} \overset{}{=} (f,\varphi) \quad \forall \varphi\in C_c^\infty(U)$$

By density argument, take $\varphi\in N(L^*)$, $(f,\varphi)=0$.

$\Longleftrightarrow f\perp N(L^*)$

Converse also holds by decomposing L^2 into kernel and co-kernel of $I-k$. ∎

Theorem (3rd Existence of weak sol.)

(i) There exist at most countable set $\Sigma\subseteq\mathbb{R}$ s.t. $\begin{cases} Lu=\lambda u+f & \text{in } U \\ u=0 & \text{on } \partial U \end{cases}$ has a unique weak solution in $H_0^1(U)$ for each $f\in L^2(U)$ iff $\lambda\notin\Sigma$.

(ii) If Σ is infinite, then $\Sigma=\{\lambda_k\}_{k=0}^\infty$ $\lambda_1\le\lambda_2\le\cdots$, $\lambda_k\to+\infty$ as $k\to\infty$.

Def. We call $\Sigma \subseteq \mathbb{R}$ the real specturm of the operator L

In particular, $\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ has a non-trivial sol. $\underset{\text{eigen function}}{\underset{\uparrow}{u = w}}$ iff $\underset{\text{eigen value}}{\underset{\uparrow}{\lambda \in \Sigma}}$.

Proof of Theorem:

- Take $\gamma > 0$ as the same const. as before. Assume $\lambda > -\gamma$.

$$\begin{cases} Lu = \lambda u + f & \text{in } U. \\ u = 0 & \text{on } \partial U \end{cases} \xrightarrow[\text{Alternative}]{\text{Fredholm}} \text{has a unique sol. for each } f \in L^2(U) \quad \text{iff} \quad \begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } U \end{cases} \text{has only trivial weak sol.}$$

$$\Leftrightarrow \begin{cases} \underbrace{Lu + \gamma u}_{L_\gamma u \text{. invertible}} = (\lambda + \gamma) u & \text{in } U. \\ u = 0 & \text{on } \partial U \end{cases} \Leftrightarrow u = L_\gamma^{-1}\big((\lambda+\gamma)u\big) = \frac{\lambda+\gamma}{\gamma} Ku.$$

Recall: $K : L^2 \to L^2$ is bounded linear compact operator

Q: Is $u \equiv 0$ the only sol. of $Ku = \frac{\gamma}{\gamma+\lambda} u$?

Ans: It happens iff $\frac{\gamma}{\lambda+\gamma}$ is NOT an eigen-value of K.

Hence it suffices to consider eigenvalues of K. From general theory of compact operators, we know eigenvalues of K is (i) either a finite set

(ii) or a sequence converging to zero. $\Leftrightarrow \lambda_n \to +\infty$.

Theorem (Boundedness of Inverse)

If $\lambda \notin \Sigma$, $\|u\|_{L^2(U)} \le C \|f\|_{L^2(U)}$ $\forall f \in L^2(U)$. $u \in L^2$ is the unique weak sol. of

$$\begin{cases} Lu = \lambda u + f & \text{in } U. \\ u = 0 & \text{on } \partial U \end{cases}$$

$C(\lambda, U, \text{coefficients of } L)$

$C \to \infty$ if $\lambda \to$ eigenvalue.

Proof Assume not. Then $\exists \{f_k\}_{k=1}^\infty \in L^2$ $\{u_k\}_{k=1}^\infty \in H_0^1(U)$ s.t. $\begin{cases} Lu_k = \lambda u_k + f_k & \text{in } U. \\ u_k = 0 & \text{on } \partial U \end{cases}$

But $\|u_k\|_{L^2} > k \|f_k\|_{L^2}$ $\forall k$.

WLOG. Assume $\|u_k\|_{L^2} = 1 \Rightarrow \|f_k\|_{L^2} < \frac{1}{k} \longrightarrow 0$ as $k \to \infty$.

Also, by Energy Estimate $\sup \|u_k\|_{H_0^1(U)} \le$ const. is bounded.

$\Rightarrow \exists$ a subsequence $\begin{cases} u_{k_j} \longrightarrow u \in H_0^1(U) & \text{weakly} \\ u_{k_j} \longrightarrow u & \text{strongly in } L^2 \end{cases}$ (Rellich - Kondrachov compact embedding)

$$\begin{cases} Lu_k = \lambda u_k + f_k & \text{in } U \\ u_k = 0 & \text{on } \partial U \end{cases} \xrightarrow{k\to\infty} \begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

u is a non-trivial ($\|u\| = 1$) weak sol.

But $\lambda \notin \Sigma$. Contradiction!

Regularity: $Lu = f$, try to show u is smooth.

Motivation: Consider $-\Delta u = f$, $u \in C^\infty$, u decays suff. fast as $|x| \to \infty$.

$$\int_{\mathbb{R}^n} f^2\,dx = \int_{\mathbb{R}^n} (\Delta u)^2\,dx = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} u_{x_i x_i} u_{x_j x_j}\,dx$$

$$= -\sum \int_{\mathbb{R}^n} u_{x_i x_i x_j} u_{x_j}\,dx$$

$\|D^2 u\|_{L^2} = \|f\|_{L^2}$

$$= \sum \int_{\mathbb{R}^n} u_{x_i x_j} u_{x_i x_j}\,dx$$

$$= \int_{\mathbb{R}^n} |D^2 u|^2\,dx < +\infty \Rightarrow u \text{ should have } H^2 \text{ regularity if } f \in L^2.$$

$\Rightarrow \|D^{m+2} u\|_{L^2} \lesssim \|D^m u\|_{L^2}$

- Interior regularity

Assume $U \subseteq \mathbb{R}^n$ open, bounded. $u \in H_0^1(U)$ is a weak sol. of $Lu = f$

where $Lu = -\sum (a^{ij} u_{x_i})_{x_j} + b^i u_{x_i} + c(x) u$. (Assume uniform elliptic)

Theorem · Interior regularity

Assume $a^{ij} \in C^1(U)$, $b^i, c \in C^\infty(U)$, $f \in L^2(U)$.

Suppose $u \in H^1(U)$ is a weak sol. of $Lu = f$.

Then $u \in H^2_{loc}(U)$.