Moreover, $\forall\, V \subset\subset U$. $\|u\|_{H^2(V)} \lesssim \|f\|_{L^2} + \|u\|_{H^1(U)}$

RMK. • We do not require $u \in H_0^1(U)$.

• Since $u \in H^2_{loc}(U)$. $\underset{\sim}{Lu = f}$ a.e. in U.

previously, without regularity, $u \in H^1$. $Lu = f \in H^{-1}$ in weak sense (Integrate against a test function)

Proof: $(Lu, v) = B[u,v] = (f, v) \quad \forall\, v \in C_c^\infty(U)$

$\Rightarrow \big(\underbrace{Lu - f}_{\in L^1_{loc}}, v\big) = 0 \Rightarrow f = Lu$ a.e.

Proof of Theorem

1. Fix $V \subset\subset U$. Choose open W s.t. $V \subset\subset W \subset\subset U$.

Choose cut-off $\zeta \in C_c^\infty$ s.t. $0 \le \zeta \le 1$ $\zeta \equiv 1$ on V. $\zeta \equiv 0$ on $\mathbb{R}^n - W$.

Since u is a weak solution, (By def) $B[u,v] = (f, v) \quad \forall\, v \in H_0^1(U)$

$$\Longleftrightarrow \sum_{1\le i,j\le n} \int_U a^{ij} u_{x_i} v_{x_j} = \int_U \tilde{f} v\, dx \text{ where } \tilde{f} = f - \sum_{i=1}^n b^i u_{x_i} - cu$$

Next idea: try to plug in some v's to obtain some estimates.

Now let $h > 0$ be small. Fix $\overset{\text{direction}}{k} \in \{1, 2, \dots, n\}$. Take $v = -\underbrace{D_k^{-h}}_{\text{difference quotient}}(\zeta^2 D_k^h u)$ $\quad (D_k^h u)(x) = \dfrac{u(x + he_k) - u(x)}{h}$

Motivation. $\int a^{ij} u_{x_i} v_{x_j} =$

Put in $v = \partial_{kk} u$: $\int a^{ij} u_{x_i} (\partial_{kk} u)_{x_j} = -\int a^{ij} (\partial_k u)_{x_i} (\partial_k u)_{x_j}$ + lower order terms

$\overset{\text{Ellipticity}}{\ge} \|\partial_k D u\|_{L^2}^2$ + lower order terms

• $v = D_k^h D_k^h u$: $-\int a^{ij} (D_k^{-h} u)_{x_i} (D_k^h u)_{x_j}$ because integration by parts gives D_k^{-h}.

• $v = D_k^{-h} D_k^h u \longleftarrow$ need localization.

$$\underbrace{\sum \int a^{ij} u_{x_i} v_{x_j}}_{A} = \underbrace{\int \tilde{f} v\, dx}_{B} \qquad v = -D_k^{-h}(\zeta^2 D_k^h u)$$

$$A = -\sum \int_U a^{ij} u_{x_i} \left[D_k^{-h}(\zeta^2 D_k^h u)\right]_{x_j}$$

$$= \sum \int_U D_h^k (a^{ij} u_{x_i}) (\zeta^2 D_k^h u)_{x_j}$$

Integration by parts via Diff. quotient

$\int v D_k^{-h} w = -\int w D_k^h v$

product rule of diff. quotient.

$$= \sum \int_U a^{ij,h} \cdot (D_k^h u_{x_i}) \cdot (\xi^2 D_k^h u)_{x_j} + \int (D_k^h a^{ij}) \cdot u_{x_i} \cdot (\xi^2 D_k^h u)_{x_j}$$ where we need ξ^2 other than ξ.

D_k^h commutes with ∂_{x_j}

$$= \sum \underbrace{\int_U a^{ij,h}(D_k^h u_{x_i})(D_k^h u_{x_j})\,\xi^2}_{A_1} + \underbrace{\int a^{ij,h}(D_k^h u_{x_i})(D_k^h u)\,2\xi\xi_{x_j} + (D_k^h a^{ij})\cdot u_{x_i}\cdot\left((D_k^h u_{x_j})\xi^2 + 2\xi\xi_{x_j}(D_k^h u)\right)}_{A_2}$$

A_1 is good: by unif. Ellipticity: $A_1 \geq \theta \int_U \xi^2 |D_k^h Du|^2$

$$|A_2| \lesssim \int_U \xi \underbrace{|D_k^h Du|}_{\text{good term}} |D_k^h u| + \xi |D_k^h Du|\,|Du| + \xi |D_k^h u|\,|Du| \, dx$$ due to C^1 assumption.

Cauchy Schwartz $\leq \varepsilon \int_U \xi^2 |D_k^h Du|^2 + \frac{C}{\varepsilon}\int_W |D_k^h u|^2 + |Du|^2 \, dx$ Thanks to cut-off

Note $\int_W |D_k^h u|^2 \lesssim \int_U |Du|^2$ if h is small.

Now take $\varepsilon = \frac{\theta}{2}$.

$$|A_2| \leq \frac{\theta}{2}\int_U \xi^2 |D_k^h Du|^2 dx + C\int_U |Du|^2 dx$$

Hence $A \geq \frac{\theta}{2}\int_U \xi^2 |D_k^h Du|^2 dx - C\int_U |Du|^2 dx$

- Estimate of B.

$$|B| \leq C\int_U (|f| + |Du| + |u|)|v| \, dx$$

$\int_U |v|^2 dx \leq C\int_U |D(\xi^2 D_k^h u)|^2 dx$ since supp $v \subset\subset U$.

$$\leq C\int_W |D_k^h u|^2 + \xi^2 |D_k^h Du|^2 dx$$

$$\lesssim C\int_U |Du|^2 + \xi^2 |D_k^h Du|^2$$

Now Cauchy-Schwartz to B:

$$B \leq \varepsilon \int_U \xi^2 |D_k^h Du|^2 + \frac{C}{\varepsilon}\int |f|^2 + |u|^2 + |Du|^2$$

Choose $\varepsilon = \frac{\theta}{4}$. combining everything:

$$\int_V |D_k^h Du|^2 dx \leq \int_U \xi^2 |D_k^h Du|^2$$

$$\leq C\int_U \left(|f|^2 + |u|^2 + |Du|^2\right) \quad \forall h \text{ small}$$

By a lemma proved before of diff. quotients,

$Du \in H^1_{loc}(U; \mathbb{R}^n)$ In particular, $u \in H^2_{loc}(U)$. $\|u\|_{H^2(V)} \lesssim \|f\|_{L^2} + \|u\|_{H^1(U)}$.

2. Refinement of above estimate:

If $V \subset\subset W \subset\subset U$, $\|u\|_{H^2(V)} \lesssim \|f\|_{L^2(W)} + \|u\|_{H^1(W)}$

Now choose a new cut-off η s.t.

$$\begin{cases} \eta \equiv 1 & \text{on } W \\ \text{supp. } \eta \subseteq U \\ 0 \le \eta \le 1. \end{cases}$$

$$\sum \int a^{ij} u_{x_i} v_{x_j} \, dx = \int_U \tilde{f} v \, dx$$

Now set $v = \eta^2 u$

$$\Rightarrow LHS = \underbrace{\int_U \eta^2 |Du|}_{\geq \|Du\|^2_{L^2(W)}} \lesssim \int_U f^2 + u^2 \, dx$$

$$\Rightarrow \|u\|_{H^1(W)} \lesssim \|f\|_{L^2(U)} + \|u\|_{L^2} \Rightarrow \|u\|_{H^2(V)} \lesssim \|f\|_{L^2(U)} + \|u\|_{L^2(U)}$$

Theorem (Higher interior regularity)

Let $m \in \mathbb{N}$. Assume $a^{ij}, b^i, c \in C^{m+1}(U)$, $f \in H^m(U) = W^{m,2}(U)$

Suppose $u \in H^1(U)$ is a weak sol. of $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$ then $u \in H^{m+2}_{loc}(U)$

$\|u\|_{H^{m+2}(V)} \lesssim \|f\|_{H^m(U)} + \|u\|_{L^2(U)}$ if $V \subset\subset U$

Proof. Induction on m.

$m = 0$: Done previously.

Assume the result holds for some $m \in \mathbb{N}$. $\forall U$, coefficients $a^{ij}, b^j, c \ldots$ as above.

Now, $a^{ij}, b^i, c \in C^{m+2}(U)$ $f \in H^{m+1}(U)$ $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ WANT: $u \in H^{m+3}_{loc}(U)$

$\|u\|_{H^{m+3}(V)} \lesssim \|u\|_{L^2} + \|f\|_{H^{m+1}}$

By inductive hypothesis: $\tilde{u} \in H^{m+2}_{loc}(U)$ $\|\tilde{u}\|_{H^{m+2}(V)} \lesssim \|\tilde{u}\|_{L^2(U)} + \|\tilde{f}\|_{H^m(U)}$ under $\tilde{a}^{ij}, \tilde{b}^i, \tilde{c}, \tilde{f} \ldots$

Let α be any multi-index with $|\alpha| = m+1$.

Choose $\phi \in C_c^\infty(W)$ set $v = (-1)^{|\alpha|} D^\alpha \phi$.

$B[u, v] = (f, v)_{L^2}$

integration by parts

$\Rightarrow \; B[\bar{u}, \bar{v}] = (\bar{f}, \bar{v}) \qquad \bar{u} = D^{\alpha} u \in H^1(W)$

$$\bar{f} = D^{\alpha} f - \sum_{\beta < \alpha} \binom{\alpha}{\beta} \left[- \sum \left(D^{\alpha-\beta} a^{ij} D^{\beta} u_{x_i} \right)_{x_j} \cdots \right]$$

Since $\phi \in C_c^{\infty}(W)$, we get $\bar{u}$ is a sol. of $\begin{cases} L\bar{u} = \bar{f} \\ \bar{u} = 0 \end{cases}$ in W, $\|\bar{f}\|_{L^2} \le \|f\|_{H^{m+1}} + \|u\|_{L^2}$

Maximum Principles

Non divergence form $Lu = -\sum a^{ij} u_{x_i x_j} + \sum b^i u_{x_i} + cu$.

Assume 1) $a^{ij} = a^{ji}$.

2) $a^{ij}, b^i, c \in C^0(U)$

3) Uniform Ellipticity : $\exists\, \theta > 0$ s.t. $\sum a^{ij}(x) \xi_i \xi_j \ge \theta |\xi|^2 \quad \forall \xi \in \mathbb{R}^n$, a.e. $x \in U$.

4) U open bounded

Theorem (weak max principle) Assume $u \in C^2(U) \cap C(\bar{U})$

$Lu = -\sum a^{ij} u_{x_i x_j} + \sum b^i u_{x_i} \le 0$ (No "cu" term) on U

Prototype : $-\Delta u \le 0$ $\Rightarrow \max_{\bar{U}} u = \max_{\partial U} u$

Then $\max_{\bar{U}} u = \max_{\partial U} u$

Proof. ① First assume $Lu < 0$ in U and there exist $x_0 \in U$ s.t. $u(x_0) = \max_{x \in \bar{U}} u(x)$

Then $Du(x_0) = 0$. $D^2 u(x_0) \le 0$ (negative-defined matrix)

$$Lu\big|_{x_0} = -\sum a^{ij}(x_0)\, u_{x_i x_j}(x_0) \overset{\text{Claim}}{\ge} 0 \Rightarrow \text{Contradiction to } Lu < 0 \text{ in } U.$$

Justification: $A = (a^{ij}(x))$ is symmetric, positive-defined by assumption

$\Rightarrow A = O^T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} O$ where $O \in O(n)$: $O^T O = O O^T = I$

WLOG $x_0 = 0$. Transform Coordinate $y = OX$, then $y_0 = O x_0 = 0$.

$x = O^T y$. $\quad u_{x_i} = \sum_{k=1}^{n} u_{y_k} O_{ki} \qquad u_{x_i x_j} = \sum_{k,l} u_{y_k y_l} O_{ki} O_{lj}$

$$\sum a^{ij} u_{x_i x_j} = \sum_{k,l} \sum_{i,j} a^{ij} u_{y_k y_l} O_{ki} O_{lj} = \sum_{k=1}^{n} \lambda_k u_{y_k y_k} \ge 0.$$

Counter examples:
if domain U is not bounded.
Consider $U = \mathbb{H} := \{ x \in \mathbb{R}^n \mid x_n > 0 \}$ $\quad u(x) = x_n$.

$\Rightarrow \Delta u = 0$ in U

But $u\big|_{\partial \mathbb{H}} = 0$, $\quad \sup_{x \in \mathbb{H}} u = +\infty$.

② Now general $Lu \le 0$. use ε-trick.

$$u^{\varepsilon}(x) = u(x) + \varepsilon e^{\lambda x_1} \quad (x \in U) \quad \lambda > 0, \ \varepsilon > 0.$$

> **Corollary (Comparison)**
> If $\begin{cases} Lu \le Lv & \text{in } U \\ u \ge v & \text{on } \partial U \end{cases}$
> $\Rightarrow u \le v$ in U.

$$\Rightarrow Lu^{\varepsilon} = \underbrace{Lu}_{\le 0} + \varepsilon L(e^{\lambda x_1}) \le \varepsilon e^{\lambda x_1}(-\lambda^2 a^{11} + \lambda b^1)$$

$$a^{ij} \ge \theta \delta^{ij}$$

$$\le \varepsilon e^{\lambda x_1}(-\lambda^2 \theta + \|b\|_{\infty} \lambda) < 0 \quad \text{if take large } \lambda.$$

Then by step ①. $\max\limits_{\bar U} u^{\varepsilon} = \max\limits_{\partial U} u^{\varepsilon}$, $u^{\varepsilon} = u + \varepsilon e^{\lambda x_1}$

Send $\varepsilon \to 0^+$. $\max\limits_{\bar U} u = \max\limits_{\partial U} u$. ■

Theorem (weak max principle for $c \ge 0$) Assume $u \in C^2(U) \cap C(\bar U)$, $c \ge 0$

$$Lu = -\sum a^{ij} u_{x_i x_j} + \sum b^i u_{x_i} + cu \le 0 \quad \text{on } U$$

Then $\max\limits_{\bar U} u \le \max\limits_{\partial U} u^+$

> **Counter example:** if $c \le 0$.
> $\Omega = (0,\pi) \times (0,\pi) \times \cdots \times (0,\pi) \subseteq \mathbb{R}^n$. $u = \prod_{i=1}^{n} \sin x_i$
> $\Delta u + nu = 0 \quad (\Leftrightarrow Lu = -\Delta u - nu = 0.)$
> Now $u|_{\partial\Omega} \equiv 0$. But $\max\limits_{x \in \bar\Omega} u = 1$ when $x = (\frac{\pi}{2}, \frac{\pi}{2}, \cdots, \frac{\pi}{2})$.

RMK. if $Lu \equiv 0$ on U, then $\max\limits_{\bar U} |u| = \max\limits_{\partial U} |u|$

Proof. Let u be a sub-solution: $Lu \le 0$ Set $V = \{x \in U \mid u(x) > 0\}$.

$$Ku := Lu - cu \le -cu \quad \text{in } U$$

$$\le 0 \quad \text{in } V.$$

$\Rightarrow \max\limits_{\bar V} u = \max\limits_{\partial V} u$, by previous theorem.

$\overset{?}{=} \max\limits_{\partial U} u^+$. ■

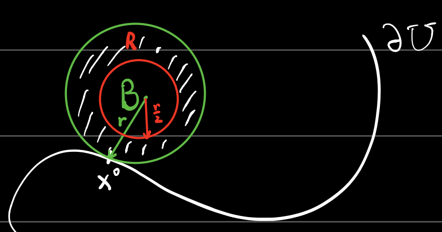
RMK. The key is to examine the extreme case $u(x_0) = \max\limits_{\bar U} u$, $Du(x_0) = 0$ $D^2u(x_0) \le 0$.

$$Lu|_{x_0} = -\sum a^{ij}(x_0) u_{x_i x_j}(x_0) + \underbrace{cu(x_0)}_{\text{want } u(x_0) > 0.}$$

Strong max principle

Lemma (Hopf) Assume $u \in C^2(U) \cap C^1(\bar U)$. and $Lu = -a^{ij} u_{x_i x_j} + b^i u_{x_i} \le 0$ (No 'cu' term)

If 1) $\exists x^0 \in \partial U$. $u(x^0) > u(x) \ \forall x \in U$.



2) U satisfies the interior ball condition at x^0
$\exists B \subseteq U$. $x^0 \in \partial B$.

Then (i) $\frac{\partial u}{\partial \nu}(x_0) > 0$ where ν is the outer unit normal vector to B.

(ii) Further if $c \ge 0$ in U then the same conclusion holds provided $u(x_0) \ge 0$

Proof. 1. Assume $c \geq 0$. $B = B(0,r)$. Define $v(x) = e^{-\lambda|x|^2} - e^{-\lambda r^2}$ $(x \in B(0,r))$ λ to be chosen (large)

$v \geq 0$.

$$Lv = -a^{ij} v_{x_i x_j} + b^i v_{x_i} + cv$$

$$= e^{-\lambda|x|^2} a^{ij} \cdot (-4\lambda^2 x_i x_j + 2\lambda \delta_{ij}) - e^{-\lambda|x|^2} b^i \cdot 2\lambda x_i + c\left(e^{-\lambda|x|^2} - e^{-\lambda r^2}\right)$$

$$\leq e^{-\lambda|x|^2}\left(-4\lambda^2\theta|x|^2 + 2\lambda\, tr(A) + 2\lambda|b||x| + c\right)$$

$|x|^2$ cannot be too small to be dominant term.

Consider the annulus $R = B(0,r) - B(0,\frac{r}{2})$.

$$Lv \leq \leq e^{-\lambda|x|^2}\left(-4\lambda^2\theta|x|^2 + 2\lambda\, tr(A) + 2\lambda|b|r + c\right) \leq 0 \quad \text{on } R \quad \text{if } \lambda \text{ is large.}$$

Recall $u(x^0) > u(x)$ $\forall x \in U$.

$\Rightarrow$ $u(x^0) \geq u(x) + \varepsilon v(x)$ if ε is small. $x \in \partial B(0,\frac{r}{2})$

Since $v(x) \equiv 0$ on $\partial B(0,r)$, we get $u(x^0) \geq u(x) + \varepsilon v(x)$ on ∂R.

$$\Rightarrow \begin{cases} L(u + \varepsilon v - u(x^0)) \leq -c\,u(x_0) \leq 0 & \text{in } R. \\ u + \varepsilon v - u(x^0) \leq 0 & \text{on } \partial R \end{cases}$$

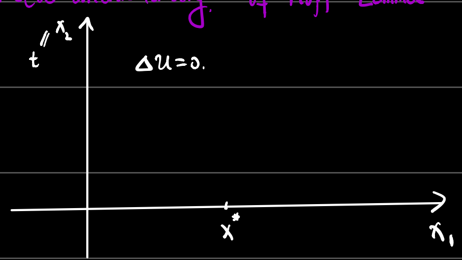
By weak max principle, $u + \varepsilon v - u(x^0) \leq 0$ in R

Now since $u(x^0) - \varepsilon v(x^0) - u(x^0) = 0$

$$\frac{\partial u}{\partial \nu}(x_0) + \varepsilon \frac{\partial v}{\partial \nu} \geq 0.$$

$$\Rightarrow \frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial v}{\partial \nu} = 2\lambda\varepsilon r e^{-\lambda r^2} > 0$$

New understanding of Hopf Lemma



$u(x) \leq u(x^*)$ $\forall x \in \mathbb{R}^2_+$

$$-\left.\frac{\partial u}{\partial t}\right|_{t=0} = |\nabla| g = C\, P.V. \int \frac{g(x^*) - g(y)}{|x_* - y|^2}\, dy$$

outer normal

$$\Delta u = (\partial_{tt} + \partial_{xx}) u = 0.$$

$u(t,x) = e^{-t|\nabla|} g$, $|\nabla| = \sqrt{-\partial_{xx}}$

boundary value.

Go back to textbook proof:

Focus on radial direction. say x^* corresponds to $r=1$

$$L \approx -\partial_{rr} - \frac{n-1}{r}\partial_r + \text{"other terms"}$$

$$\approx -\partial_{rr} \qquad \frac{1}{2} \le r \le 1$$

Look for super harmonic $g(r)$:

$g(1) = u(1)$

$g(r) > u(r) \quad \frac{1}{2} \le r < 1.$

$\Rightarrow g'(1) \le u'(1)$

eg. $g(r) = \varepsilon\left(e^{-\lambda} - e^{-\lambda r}\right), \ \lambda \gg 1.$

Theorem (Strong Max Principle) Assume $u \in C^2(U) \cap C(\bar{U})$. U is open bounded.

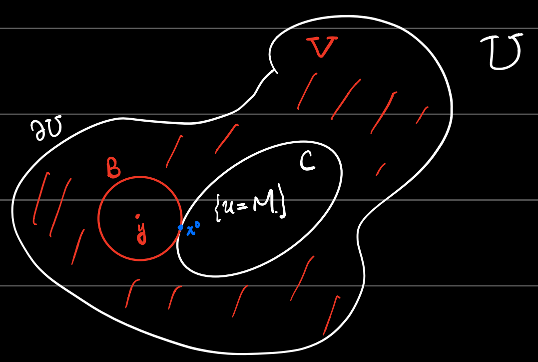
$$Lu = -a^{ij} u_{x_i x_j} + b^i u_{x_i} \le 0. \quad (\text{No "}cu\text{" term}) \quad \text{in } U$$

And suppose u attains its max over $\bar{U}$ at an interior pt, then u is const. on U.

Proof: $M := \max_{x \in \bar{U}} u$. $C = \{x \in U \mid u(x) = M\}$ Try to show $U = C$.

Assume $u \not\equiv M$. $V := \{x \in U \mid u(x) < M\}$.

Choose $y \in V$ s.t. $\operatorname{dist}(y, C) < \operatorname{dist}(y, \partial U)$. Denote by B the largest ball with center y whose interior lies in V.



$u(x^0) = M$. By Hopf Lemma, $\frac{\partial u}{\partial n}\Big|_{x^0} > 0$, n is the outer normal.

But $u(x^0)$ is max, $\nabla u(x_0) = 0$. Contradiction!

Harnack inequality

$Lu = -a^{ij}u_{x_ix_j} + b^i u_{x_i} + cu$

Theorem (Harnack inequality) Assume $u \in C^2(U)$, $Lu = 0$ in U. $u \geq 0$

Let $V \subset\subset U$ be connected. Then

$$\sup_V u \leq C_{V,L} \inf_V u$$

Eigenvalues and Eigenfunctions.

$Lu = -(a^{ij}u_{x_i})_{x_j}$, $a^{ij} = a^{ji}$. uniformly Elliptic. Assume $a^{ij} \in C^\infty(\bar{U})$

Theorem. $\begin{cases} Lw = \lambda w & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$

(i) Each eigenvalue of L is real

(ii) $\Sigma := \{\lambda_k\}_{k=1}^\infty$ (repeat each eigenvalue according to its finite multiplicity)

$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$

$\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$

(iii) There exists an orthonormal basis $(w_k)_{k=1}^\infty$ of L^2, where

$$\begin{cases} Lw_k = \lambda w_k & \text{in } U. \\ w_k = 0 & \text{on } \partial U \end{cases}$$

RMK. $w_k \in C^\infty(U)$. Further $w_k \in C^\infty(\bar{U})$ if ∂U is smooth.

Proof. $S = L^{-1}$ is a bounded linear compact operator mapping $L^2(U)$ into itself.

S is symmetric. Then apply general theory of symmetric compact operator. ∎

More generally, consider $Lu = -a^{ij}u_{x_ix_j} + b^i u_{x_i} + cu$.

Assume • $a^{ij} = a^{ji}, b, c \in C^\infty(\bar{U})$

• U open bounded, connected. ∂U is smooth.

- $c \geq 0$

in general $L \neq$ its formal adjoint.

Theorem. (Principal eigenvalue for non-symmetric elliptic operator)

(i) There exists a real eigenvalue $\lambda_1 \geq 0$ for L: $\begin{cases} Lu = \lambda_1 u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ s.t. $\lambda \in \mathbb{C}$ any other eigenvalue, then

$\operatorname{Re}(\lambda) \geq \lambda_1$

(ii) $\begin{cases} Lw_1 = \lambda_1 w_1 & \text{in } U \\ w_1 = 0 & \text{on } \partial U \end{cases}$ $w_1 > 0$ in U

(iii) λ is simple: $\begin{cases} Lu = \lambda_1 u \\ u = 0 \end{cases} \Rightarrow u = c \cdot w_1$ for some const. c.

RMK. The proof uses positivity in an essential way and can be generalized.