Examples of normed vector spaces

—— Functional Analysis

Example 1 $\ell^\infty = \left\{ (x_n)_{n=1}^\infty \;\middle|\; \sup_{n\in\mathbb{N}} |x_n| < +\infty,\ x_n \in \mathbb{C} \right\}$

with the norm

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$$

Theorem 1 $(\ell^\infty, \|\cdot\|_\infty)$ is complete.

proof. Take a sequence of sequences in ℓ^∞:

$x^1, x^2, x^3, \cdots$ Cauchy.

each $x^n = (x^n_k)_{k=1}^\infty$

Cauchy $\Leftrightarrow$ $\forall \varepsilon > 0.\ \exists n_0.\ \forall m, n \geq n_0$

$$\|x^n - x^m\|_\infty = \sup_j \|x^n_j - x^m_j\| \leq \varepsilon$$

In particular, for each fixed j,

$(x^n_j)_{n=1}^\infty$ is Cauchy in $\mathbb{C}$.

Hence $\exists\, x_j \in \mathbb{C}.\ x^n_j \to x_j$ as $n \to \infty$.

Call $x := (x_j)_{j=1}^{\infty}$

Claim: $x^n \to x$ in ℓ^∞.

$$\|x^n - x\|_\infty = \sup_j \|x_j^n - x_j\| \xrightarrow[\text{by construction.}]{} 0 \text{ as } n \to \infty$$

$x \in \ell^\infty$, since

$$|x_j| = \lim_{n\to\infty} |x_j^n| \le \limsup_{n\to\infty} \|x^n\| \le C_0.$$

Cauchy $\Rightarrow$ bounded.

Take sup over j. $x \in \ell^\infty$.

$\square$

Example 2. $\ell^p = \{ (x_n)_{n=1}^{\infty} \mid x_n \in \mathbb{C},\ \sum_{n=1}^{\infty} |x_n|^p < +\infty \}$

with norm $\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$

RMK. Inequalities:

① Hölder $\|xy\|_1 \le \|x\|_p \|y\|_q$

$\frac{1}{p} + \frac{1}{q} = 1$. $1 \le p, q \le \infty$.

② Minkowski.

$\|x+y\|_p \le \|x\|_p + \|y\|_p$

Theorem 2 $(\ell^p, \|\cdot\|_p)$ is complete.

proof of Theorem 2: Given a Cauchy sequence $(x^n)_{n=1}^{\infty}$ in ℓ^p:

$\forall \varepsilon > 0. \quad \exists n_0. \forall n, m \geq n_0$

$$\|x^n - x^m\|_p^p = \left|\sum_{j\geq 1} x_j^n - x_j^m\right|^p \leq \varepsilon.$$

$\Rightarrow$ $|x_1^n - x_1^m|^p \leq \varepsilon$

$(x_1^n)_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$.

$\exists x_1 \in \mathbb{C}$. $x_1^n \to x_1$.

Similarly, $\forall j$, $\exists x_j \in \mathbb{C}$. $x_j^n \to x_j$

Now define $x = (x_1, x_2, \cdots)$

WANT: $\|x\|_p < +\infty$.

Fix $N \in \mathbb{N}$.

$$\sum_{n=1}^{N} |x_n|^p = \lim_{n\to\infty} \sum_{j=1}^{N} |x_j^n|^p$$

$$\leq \lim_{n\to\infty} \underbrace{\sum_{j=1}^{\infty} |x_j^n|^p}$$

$$= \limsup_{n\to\infty} \|x^n\|_p^p$$

$\leq C_0$ since (x^n) is Cauchy in ℓ^p

Letting $N \to \infty$, we get $\|x\|_p < +\infty$,

$x \in \ell^p$.

Now show $x^n \to x$ in ℓ^p:

Since (x^n) is Cauchy in ℓ^p,

$\forall \varepsilon > 0$, $\exists n_0$, if $m, n > n_0$:

$$\|x^m - x^n\|_p^p \le \varepsilon$$

$$\sum_{j=1}^{\infty} |x_j^n - x_j^m|^p \le \varepsilon$$

$$\Rightarrow \sum_{j=1}^{N} |x_j^n - x_j^m|^p \le \varepsilon. \quad \forall N.$$

As $x_j = \lim_{m\to\infty} x_j^m$, sending $m \to \infty$.

$$\sum_{j=1}^{N} |x_j^n - x_j^m|^p \le \varepsilon.$$

RHS is independent of N,

hence sending $N \to \infty$ will do the job.

$\square$

Example 3. (M, d) metric space

$$C_c(M) := \{ f : M \to \mathbb{C} \text{ continuous with compact support} \}$$

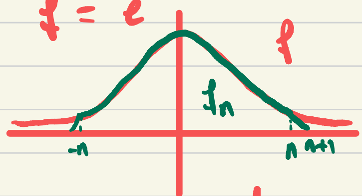
with norm

$$\|f\|_\infty := \sup_{x \in M} |f(x)|$$

RMK. $C_c(M)$ is NOT complete.

Eg. $M = \mathbb{R}$, $f = e^{-x^2}$

$f_n =$



limit may not have compact support.

However, if M is compact, then

$C(M)$ is complete.

Example 4.

Take $\Omega \subseteq \mathbb{R}^n$ open connected.

$$C_c(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ cts, compact supported} \right\}$$

NOT complete if $\|\cdot\|_\infty$.

Define a new norm

$$\|f\|_p := \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}, \quad p \geq 1$$

is NOT complete.

(ℓ^p limit may not be continuous)