(Finite product measures and Fubini)

We start with $(\Omega_j, \mathcal{F}_j, \mu_j)$ $j = 1,2$. measure spaces.

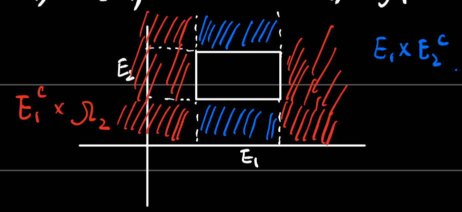
Rectangles: $E := E_1 \times E_2$ where $E_j \in \mathcal{F}_j$. $\mathcal{F}_1 \times \mathcal{F}_2$:= set of all rectangles.

Claim 1. $\mathcal{F}_1 \times \mathcal{F}_2$ is an Algebra.

proof: (i) $\phi \times \phi$, $\Omega_1 \times \Omega_2 \in \mathcal{F}_1 \times \mathcal{F}_2$. trivial.

(ii) Stable under finite intersections: $A_1 \times B_1$, $A_2 \times B_2 \in \mathcal{F}_1 \times \mathcal{F}_2$. Then $(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2) \in \mathcal{F}_1 \times \mathcal{F}_2$.

(iii) 补集: 对 $E_1 \times E_2 \in \mathcal{F}_1 \times \mathcal{F}_2$: $(E_1 \times E_2)^c = E_1^c \times \Omega_2 \cup E_1 \times E_2^c$



∎

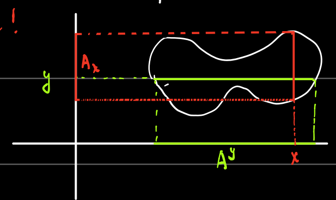
Def. The sigma-algebra generated by $\mathcal{F}_1 \times \mathcal{F}_2$: $\sigma(\mathcal{F}_1 \times \mathcal{F}_2) := \mathcal{F}$. Define the product pre-measure $\mu: \mathcal{F}_1 \times \mathcal{F}_2 \to [0, +\infty]$ by $\mu(E_1 \times E_2) := \mu_1(E_1) \times \mu_2(E_2)$. with the convention $0 \cdot \infty = 0$.

Claim 2. μ is σ-additive on $\mathcal{F}_1 \times \mathcal{F}_2$. Hence by Caratheodory extension Thm, we can extend μ onto $\mathcal{F} := \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$

proof: (1) μ is additive on $\mathcal{F}_1 \times \mathcal{F}_2$.

*Lemma 1. If $A \in \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$. $\underset{\text{cross section}}{A_x} := \{ y \in \Omega_2 \mid (x,y) \in A \}$ $A^y = \{ x \in \Omega_1 \mid (x,y) \in A \}$

Important!



Then $A_x \in \mathcal{F}_2$. $\forall x \in \Omega_1$. $A^y \in \mathcal{F}_1$ $\forall y \in \Omega_2$

？与原定 Corollary 3.3 的矛盾: 原定对 a.e $x \in \Omega_1$ 成立.

Proof of the Lemma 1: $\mathcal{C} := \{ A \subseteq \Omega_1 \times \Omega_2 \mid A \text{ satisfies the above property} \}$

(i) $\mathcal{F}_1 \times \mathcal{F}_2 \subseteq \mathcal{C}$: if $E = E_1 \times E_2 \in \mathcal{F}_1 \times \mathcal{F}_2$. $E_x = \begin{cases} \phi & \text{if } x \notin E_1 \\ E_2 & \text{if } x \in E_1 \end{cases}$ So $E_x \in \mathcal{F}_2$.

(ii) $\mathcal{C}$ is a σ-algebra:

(ii.1) $\Omega := \Omega_1 \times \Omega_2 \in \mathcal{C}$: trivial since $\Omega \in \mathcal{F}_1 \times \mathcal{F}_2$.

(ii.2) 补集: Suppose $A \in \mathcal{C}$. Fix $x \in \Omega_1$. $A_x \in \mathcal{F}_2$. $(A^c)_x = \{ y \in \Omega_2 \mid (x,y) \in A^c \} = \{ y \in \Omega_2 \mid (x,y) \notin A \} = (A_x)^c \in \mathcal{F}_2$.

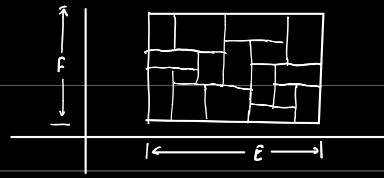
(ii.3) Countable union: Suppose $\{A_j\}_{j=1}^{\infty} \in \mathcal{C}$: Fix $x \in \Omega_1$. $(A_j)_x \in \mathcal{F}_2$.

$(\bigcup A_j)_x = \{ y \in \Omega_2 \mid (x,y) \in \bigcup A_j \} = \bigcup_{j \geq 1} \{ y \in \Omega_2 \mid (x,y) \in A_j \} = \bigcup_{j \geq 1} (A_j)_x \in \mathcal{F}_2$.

$\Rightarrow \mathcal{C}$ is an σ-algebra containing $\sigma(\mathcal{F}_1 \times \mathcal{F}_2)$. but $\mathcal{C} \subseteq \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$. Hence they are equal.

This completes the proof of Lemma 1. ∎

Now return to additivity of $\mu := \mu_1 \times \mu_2$ on $\mathcal{F}_1 \times \mathcal{F}_2$.



Suppose $A = \biguplus_{j=1}^{n} A_j$ with $A_j = E_j \times F_j \in \mathcal{F}_1 \times \mathcal{F}_2$. Write $A = E \times F$.

$A_x = \begin{cases} \emptyset & x \notin E \\ F & x \in E. \end{cases}$ $\quad A_x = \left(\biguplus_{j=1}^{n} A_j\right)_x = \biguplus_{j=1}^{n} (A_j)_x = \begin{cases} \biguplus_{j=1}^{n} F_j \mathbb{1}_{E_j} & 若\ x \in E \text{ in some } E_j. \\ \emptyset & 若\ x \notin E. \end{cases}$

$\Rightarrow F = \biguplus_{j=1}^{n} F_j \mathbb{1}_{E_j}$ $\quad \mathbb{1}_E \, \mu_2(F) = \sum_{j=1}^{n} \mu_2(F_j) \mathbb{1}_{E_j}$ for $x \in$ some E_j

两边积分(对μ_1): $\mu_1(E)\mu_2(F) = \sum_{j=1}^{n} \mu_2(F_j)\mu_1(E_j)$. Add- is proved.

(2) σ- add. n 改成 ∞. 最后积分那一步用 MCT 即可. 其余 argument 仍成立.

(3) σ- finiteness of Ω: Suppose Ω_j is σ-finite: $\exists \{A_n^{(j)}\}_{n=1}^{\infty}$ $\Omega_j = \bigcup_{n=1}^{\infty} A_n^{(j)}$, $\mu_j(A_n^{(j)}) < +\infty$ W.L.O.G. assume $A_n^{(j)} \uparrow$.

Then $A_n := A_n^{(1)} \times A_n^{(2)}$, $\bigcup_{n=1}^{\infty} A_n = \Omega$ and $\mu(A_n) = \mu_1(A_n^{(1)})\mu_2(A_n^{(2)}) < +\infty$. $\Rightarrow$ μ is σ-finite on Ω.

By Caratheodory, the extension of μ onto $\mathcal{F}$ is unique.

We then finish the construction of product measure. ∎

Thm. (Fubini) Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be σ-finite. Let $f \in \sigma(\mathcal{A} \times \mathcal{B})$. then

(a) (Tonelli) If $f \geq 0$, $\varphi(x) = \int_Y f(x,y)\, d\nu$, $\psi(y) = \int_X f(x,y)\, d\mu$. then

$\varphi \in \mathcal{A}$. $\psi \in \mathcal{B}$ are measurable functions. and 对非负函数 $\int_{X\times Y} |f(x,y)|\, d(\mu\times\nu) < +\infty$ 不必要

$$\int_X \varphi(x)\, d\mu = \int_Y \psi(y)\, d\nu = \int_{X\times Y} f(x,y)\, d(\mu\times\nu)$$

(b) if f is complex-valued: $\varphi^*(x) = \int_Y |f|\, d\nu$, $\psi^*(y) = \int_X |f|\, d\mu$

If $\int_Y \psi^* d\nu$ and $\int_X \varphi^* d\mu$ are finite then $f \in L^1(\mu\times\nu)$. $\int_X \varphi(x)\, d\mu = \int_Y \psi(y)\, d\nu = \int_{X\times Y} f(x,y)\, d(\mu\times\nu)$

R.M.K.

1) σ-finiteness is critical. Counter-example: Let μ be Lebesgue measure on $[0,1]$, ν be counting measure on $[0,1]$. then Fubini does not holds.

2) f must be measurable w.r.t. product measure.

3) In the general case (b). $\int_X \left(\int_Y |f(x,y)|\, d\nu\right) d\mu < +\infty$ is needed. Otherwise it could happen such that $\int_Y \left(\int_X f\, d\mu\right) d\nu < +\infty$. $\int_X \int_Y f\, d\nu\, d\mu < +\infty$ but they are NOT equal.

Proof. (a) Suppose $f \in \sigma(\mathcal{A}\times\mathcal{B})$: approximate f by $\{f_n\}_{n=1}^{\infty}$ simple. $f_n \uparrow f$. f_n simple then use MCT.

(b) $f = f_1 + i f_2 = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$. Approximate $f_i^{\pm}$ by simple functions. done. ∎

Corollary. Can be generalized to finite product measure.

详细证明. (Ref. IMPA)

Lemma 2. $f: \Omega_1 \times \Omega_2 \longrightarrow \overline{\mathbb{R}}$. f is $\mathcal{F}$-measurable. $\forall x \in \Omega_1$: $f_x(y): \Omega_2 \longrightarrow \overline{\mathbb{R}}$. $f_x(y) := f(x,y)$ is $\mathcal{F}_2$-measurable.

Proof: Take a borel set $B \subseteq \overline{\mathbb{R}}$. $f_x^{-1}(B) = \{y \in \Omega_2 \mid f(x,y) \in B\} = \big(\underset{\underset{\mathcal{F}}{\cup}}{f^{-1}(B)}\big)_x$ is $\mathcal{F}_2$-measurable by Lemma 1 above. ∎

Lemma 3. $E \in \mathcal{F} =: \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$. Claim: $x \xmapsto{\varphi} \mu_2(E_x)$ is $\mathcal{F}_1$-measurable and $y \xmapsto{\psi} \mu_1(E^y)$ is $\mathcal{F}_2$-measurable.

(1) Assume $\mu_1(\Omega_1), \mu_2(\Omega_2) < +\infty$. (1.i) Assume $E \in \mathcal{F}_1 \times \mathcal{F}_2$. $E_x = \begin{cases} E_2 & \text{if } x \in E_1 \\ \phi & \text{if } x \notin E_1 \end{cases}$. $\mu_1(E_x) = \mathbb{1}_{E_1}(x)\mu_2(E_2)$ is a simple function of x thus $\mathcal{F}_1$-measurable

(1.ii) Assume E finite disjoint union of rectangles $E = \bigsqcup_{j=1}^{n} E_j = \bigsqcup_{j=1}^{n} A_j \times B_j$

$E_x = \left(\bigsqcup_{j=1}^{n} E_j\right)_x = \bigsqcup_{j=1}^{n} (E_j)_x = \bigsqcup_{j=1}^{n} B_j \mathbb{1}_{A_j}(x)$. By Lemma 1, $E_x \in \mathcal{F}_2$.

$\mu_2(E_x) = \mu_2\left(\bigsqcup_{j=1}^{n} B_j \mathbb{1}_{A_j}(x)\right) = \sum_{j=1}^{n} \mu_2(B_j)\mathbb{1}_{A_j}(x)$ simple function of x. hence $\mathcal{F}_1$-measurable.

(1.iii) $\mathcal{G} = \{E \in \mathcal{F} \mid \mu_2(E_x) \text{ is } \mathcal{F}_1\text{-measurable}\}$. $\mathcal{F}_1 \times \mathcal{F}_2 \subseteq \mathcal{G}$. Now prove $\mathcal{G}$ is a σ-algebra hence $\mathcal{G} = \mathcal{F}$.:

Using Monotone Class Thm. Claim: $\mathcal{G}$ is a monotone class.

① $\{E^{(n)}\}_{n=1}^{\infty} \in \mathcal{G}$. $E^{(n)} \uparrow E$. $\mu_2(E_x^{(n)})$ is $\mathcal{F}_1$-measurable. $E_x^{(n)} \nearrow E_x$, $\forall x \in \Omega_1$. $\Rightarrow$ $\mu_2(E_x^{(n)}) \nearrow \mu_2(E_x^{(n)})$ since μ_2 is a measure.

Being a limit of measurable functions. $x \longmapsto \mu_2(E_x)$ is $\mathcal{F}_1$-measurable.

② $\{E^{(n)}\}_{n=1}^{\infty} \downarrow E$. $E_x^{(n)} \downarrow E_x$. Since $\mu_2(\Omega_2) < +\infty$, $\mu_2(E_x^{(n)}) \downarrow \mu_2(E_x)$

$\xRightarrow[\text{Thm.}]{\text{Monotone Class}}$ $\mathcal{G}$ is a σ-algebra since $\sigma(\mathcal{F}_1 \times \mathcal{F}_2) = \mathcal{M}(\mathcal{F}_1 \times \mathcal{F}_2)$.

(2) Assume Ω_1, Ω_2 are σ-finite. $\exists A_n, B_n \uparrow \Omega_1, \Omega_2$. $\mu_1(A_n), \mu_2(B_n) < +\infty$

$\mu_2(E_x) = \lim_{n \to \infty} \mu_2(E_x \cap B_n)$. But $x \longmapsto \mu_2(E_x \cap B_n)$ is $\mathcal{F}_1$-measurable by (1).

Being the limit of measurable functions, $x \longmapsto \mu_2(E_x)$ is $\mathcal{F}_1$-measurable.

This completes the proof of Lemma 3. ∎

Lemma 4. $\int \mu_2(E_x)\, d\mu_1 = \mu(E) = \int \mu_1(E^y)\, d\mu_2$ (*). $\forall E \in \mathcal{F}$.

(i) $E \in \mathcal{F}_1 \times \mathcal{F}_2$. $E = A \times B$. $\mu_2(E_x) = \mu_2(B)\mathbb{1}_A(x)$. 对 x 积分之，证毕.

(ii) $E = \bigsqcup_{j=1}^{n} E_j$. $E_j \in \mathcal{F}_1 \times \mathcal{F}_2$: $\mu_2(E_x) = \sum_{j=1}^{n} \mu(B_j)\mathbb{1}_{A_j}(x)$ 对 x 积分之，证毕.

(iii) $E = \bigsqcup_{j=1}^{\infty} E_j$. Monotone convergence Thm. 证毕. ∎

Thm (Tonelli) $f \geq 0$. $\mathcal{F}$-measurable. then $\varphi(x) = \int_{\Omega_2} f(x,y)\, d\mu_2$ is $\mathcal{F}_1$-measurable, integrable. and.

$$\int \Big(\underbrace{\int f_x(y)\, d\mu_2}_{= \varphi(x)}\Big)\, d\mu_1 = \int f(x,y)\, d\mu.$$

Proof.

Characteristic functions: $f = \mathbb{1}_E$. $f_x(y) = \mathbb{1}_{E_x}(y)$ $\int f_x(y)\, d\mu_2 = \mu_2(E_x)$ is measurable by Lemma 3.

and $\int \left(\int \mathbb{1}_E\, d\mu_2\right) d\mu_1 = \int \mu_2(E_x)\, d\mu_1 = \mu(E)$ by Lemma 4 above.

Using Linearity, extend to simple functions.

Now approximate f by simple functions and apply MCT. ∎

Thm. (Fubini) Change "$f \geq 0$" by "$\int f\, d\mu < +\infty$". in Tonelli

Proof. Decompose $f = f^+ - f^-$. By Tonelli. $\int \Big(\underbrace{\int f_x^+(y)\, dy}_{:= \varphi(x)\ \mathcal{F}_1\text{-measurable.}}\Big) dx = \int f^+\, d\mu$

$\int f^{\pm} d\mu < +\infty$. $\Rightarrow$ $\varphi^{\pm}(x) = \int f_x^{\pm}(y)\, dy < +\infty$ a.e. Replace $\tilde{\varphi}^{\pm}(x) = \begin{cases} \varphi^{\pm}(x) & \text{if } \varphi^{\pm}(x) < +\infty \\ 0 & \text{otherwise.} \end{cases}$ $\mathcal{F}_1$-measurable

Similarly. $\tilde{\psi}^{\pm} = \begin{cases} \psi^{\pm}(y) & \text{if } \psi^{\pm}(y) < +\infty \\ 0 & \text{otherwise.} \end{cases}$ Now $\tilde{\varphi}^+(x) - \tilde{\varphi}^-(x)$ makes sense for all x.

Now by Tonelli. $\int \tilde{\varphi}^{\pm} d\mu_1 = \int f^{\pm} d\mu$ and hence $\int \tilde{\varphi}^+ - \tilde{\varphi}^-\, d\mu_1 = \int f\, d\mu$.

Infinite product measure space? Kolmogorov Extension Thm.

(Gentle intro to Brownian Motion and Brownian measure)

Brownian motion: Definitions first.

Def. 1) Stochastic Process. $T=[0,+\infty)$ $X_t = X_t(\omega): (\Omega, \mathcal{F}, P) \to \mathbb{R}$ is called a Sto. Pro. if for each $t \in T$, X_t is a r.v.

2) Finite dimensional distribution. $t_1, t_2, \dots, t_k \in [0,+\infty)$

$\mu_{t_1,\dots,t_k}(F_1 \times \dots \times F_k) = P(X_{t_1} \in F_1, X_{t_2} \in F_2, \dots, X_{t_k} \in F_k)$ where F_i are borel sets on $\mathbb{R}$.

Consistency: Note $\mu_{t_{\tau(1)},\dots,t_{\tau(k)}}(F_{\tau(1)} \times \dots \times F_{\tau(k)}) = \mu_{t_1,\dots,t_k}(F_1 \times \dots \times F_k)$ where $\tau \in S_k$ is any permutation.

Marginal: $\mu_{t_1,\dots,t_k}(F_1 \times \dots \times F_k) = \mu_{t_1,\dots,t_k,t_{k+1}}(F_1 \times \dots \times F_k \times \mathbb{R})$

This way we have a family of consistent finite dimensional distributions.

Question: 已知$\{\mu_{t_1,\dots,t_k}\}$，是否存在对应的 stochastic process?

Ans: Yes. Kolmogorov Extension Thm.

Notations. Let $\vec{t} = (t_1, \dots, t_n)$ be distinct non-negative numbers. Suppose for each $\vec{t}$ of length n, we have a probability measure $Q_{\vec{t}}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

Thm (Daniell 1918, Kolmogorov 1933)

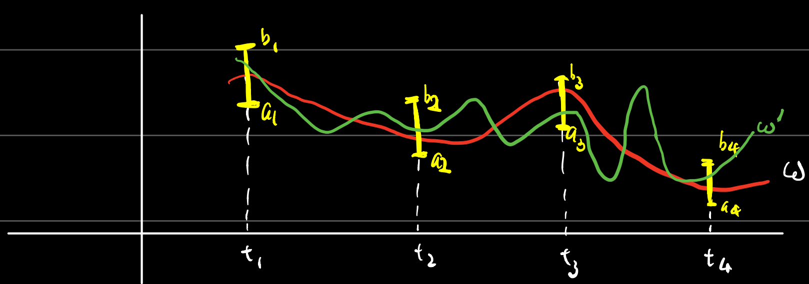
Let $\{Q_{\vec{t}}\}$ be a family of consistent finite dimensional distribution. Then $\exists$ a probability measure P on $(\mathbb{R}^{[0,+\infty)}, \mathcal{B}([0,+\infty))$ s.t. $Q_{\vec{t}}(A) = P(\omega \in \mathbb{R}^{[0,+\infty)} : (\omega(t_1), \dots, \omega(t_n)) \in A$. where $A \in \mathcal{B}(\mathbb{R}^n))$

$\mathbb{R}^{[0,+\infty)}$:= the set of all real-valued functions on $[0,+\infty)$

$\omega \in \mathbb{R}^{[0,+\infty)}$ looks like. $[0,+\infty) \to \mathbb{R}$

Cylinder set. $A \in \mathcal{B}(\mathbb{R}^n)$ $C := \{\omega \in \mathbb{R}^{[0,+\infty)} : (\omega(t_1), \dots, \omega(t_n)) \in A\}$

eg. $A = \times_{i=1}^{n} [a_i, b_i]$, then $C = \{\omega \in \mathbb{R}^{[0,+\infty)} : a_i \le \omega(t_i) \le b_i, \forall i.\}$



$\mathcal{C}$: the field generated by all cylinder sets (of all finite dimension distribution) in $\mathbb{R}^{[0,+\infty)}$

$\mathcal{B}(\mathbb{R}^{T}) := \sigma$- algebra generated by $\mathcal{C}$

Proof. Step 1) Define a set function $\mathbb{P}: \mathcal{C} \to [0,+\infty]$.

$\mathbb{P}(C) := Q_{\vec{t}}(A)$. $\forall\, C \in \mathcal{C}$.

One can show $\mathbb{P}$ on $\mathcal{C}$ is well defined and finitely addtive and $\mathbb{P}(\mathbb{R}^{[0,+\infty)}) = 1$.

Step 2) Extend $\mathbb{P}$ onto $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^{[0,+\infty)})$ by Caratheodory extension Thm.

Need to check countable additivity of $\mathbb{P}$ on $\mathcal{C}$.

Suppose $\{B_k\}_{k=1}^{\infty}$ disjoint sets in $\mathcal{C}$ with $B := \bigcup_{n=1}^{\infty} B_n$ also in $\mathcal{C}$. Goal: $\mathbb{P}(B) = \sum_{k=1}^{\infty} \mathbb{P}(B_k)$.

Denote $C_m = B \setminus \bigcup_{n=1}^{m} B_n = \bigcup_{n=m+1}^{\infty} B_n$. "tail sets".

$$\mathbb{P}(B) = \mathbb{P}(C_m) + \mathbb{P}\left(\bigcup_{n=1}^{m} B_n\right)$$

finite add.

$$= \mathbb{P}(C_m) + \sum_{n=1}^{m} \mathbb{P}(B_n)$$

Countable add. $\Longleftrightarrow \lim_{m\to\infty} \mathbb{P}(C_m) = 0$.

Step 3) $\{C_m\}_{m=1}^{\infty} \downarrow \phi$. $\mathbb{P}(C_m) = \mathbb{P}(C_{m+1}) + \mathbb{P}(B_m)$ So $\mathbb{P}(C_m)\downarrow$.

Assume on the contrary $\lim_{m\to+\infty} \mathbb{P}(C_m) = \varepsilon > 0$.

From $\{C_m\}_{m=1}^{\infty}$ we construct $\{D_m\}_{m=1}^{\infty}$ s.t. $D_1 \supseteq D_2 \supseteq \cdots$, $\bigcap_{m=1}^{\infty} D_m = \phi$. $\lim \mathbb{P}(D_m) = \varepsilon > 0$.

$D_m = \{\omega \in \mathbb{R}^{[0,+\infty)} : (\omega(t_1), \dots, \omega(t_m)) \in A_m\}$ for some $A_m \in \mathcal{B}(\mathbb{R}^n)$, $\vec{t}_m = (t_1, \dots, t_m)$

Idea of Construction: $C_k = \{\omega \in \mathbb{R}^{[0,+\infty)} : (\omega(t_1), \dots, \omega(t_{m_k})) \in A_{m_k}\}$

Since $C_{k+1} \subseteq C_k$. Choose a representations s.t. $\vec{t}_{m_{k+1}}$ is an extension (adding more points) of $\vec{t}_{m_k}$, and $A_{m_{k+1}} \subseteq A_{m_k} \times \mathbb{R}^{m_{k+1}-m_k}$.

Define $D_1 = \{\omega \in \mathbb{R}^{[0,+\infty)} : \omega(t_1) \in \mathbb{R}\}$

$\vdots$

$D_{m_1-1} = \{\omega : (\omega(t_1), \cdots \omega(t_{m_1-1})) \in \mathbb{R}^{m_1-1}\}$

$D_{m_1} = C_1 = \{\omega : (\omega(t_1), \cdots \omega(t_{m_1})) \in A_1\}$

Next idea: $\lim \mathbb{P}(D_m) = \varepsilon > 0$ contradicts $\bigcap D_m = \phi$.

Def regular set: $A \in \mathcal{B}(\mathbb{R}^n)$, $\forall\, \mathbb{P}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ one can find F closed. G open, $F \subseteq A \subseteq G$.

$\mathbb{P}(G \setminus F) < \delta$. $\forall \delta$. Eg: Borel sets are regular. $(*)$

For each m, we find F_m closed. $A_m \supseteq F_m$. $Q_{\vec{t}_m}(A_m \setminus F_m) < \frac{\varepsilon}{2^m}$

Now intersect F_m with $\overline{B(0, r_m)}$ for large enough r_m to get a compact set K_m: s.t.

$E_m = \{\omega \in \mathbb{R}^{[0,+\infty)} : (\omega(t_1), \dots, \omega(t_m)) \in K_m\}$

$\mathbb{P}(D_m \setminus E_m) = Q_{\vec{t}_m}(A_m \setminus K_m) < \frac{\varepsilon}{2^m}$

But E_m may fail to be $\downarrow$. So $\tilde{E}_m := \bigcap_{n=1}^m E_n$ $\quad \tilde{K}_m = (K_1 \times \mathbb{R}^{m-1}) \cap (K_2 \times \mathbb{R}^{m-2}) \cap \dots \cap K_m$ still compact.

Claim: $Q_{\vec{t}_m}(\tilde{K}_m) > 0$.

$$\begin{aligned}
Q_{\vec{t}_m}(\tilde{K}_m) = \mathbb{P}(\tilde{E}_m) &= \mathbb{P}(D_m) - \mathbb{P}(D_m \setminus \tilde{E}_m) \\
&= \mathbb{P}(D_m) - \mathbb{P}\Big(D_m \setminus \bigcup_{k=1}^m E_k\Big) \\
&\geq \mathbb{P}(D_m) - \mathbb{P}\Big(\bigcup_{k=1}^m D_k \setminus E_k\Big) \\
&\geq \varepsilon - \sum_{k=1}^m \frac{\varepsilon}{2^k} > 0.
\end{aligned}$$

$\Rightarrow$ $\tilde{K}_m$ is non-empty for each m.

For each m, we can choose $(x_1^{(m)}, \dots, x_m^{(m)}) \in \tilde{K}_m$. Now contradiction as follows.

$\{x_1^{(m)}\}_{m=1}^\infty \in \tilde{K}_1$, must contain a conv. subsequence $\{x_1^{(m_k)}\}_{k=1}^\infty$ with limit x_1

$\{x_2^{(m_k)}\}_{k=1}^\infty$ Pick a further subsequence... with limit x_2.

$\vdots$

(Diagonalization Process)

get $(x_1, x_2, \dots) \in \mathbb{R} \times \mathbb{R} \times \cdots$ s.t. $(x_1, \dots x_m) \in K^m$ each m.

Thus $S = \{\omega \in \mathbb{R}^{[0,+\infty)} : \omega(t_i) = x_i\}$ is in each D_m, contradicting $\bigcap_{m=1}^\infty D_m = \phi$.