Def. Let X be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

A **norm** on X is a function $\|\cdot\| : X \to \mathbb{R}_+$

s.t. (i) $\|x\| \geq 0$.

(ii) $\|x\| = 0$ iff $x = 0$

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\alpha \in \mathbb{F}$.

(iv) $\|x+y\| \leq \|x\| + \|y\|$.

RMK. Triangle inequality: $\big| \|x-y\| \big| \leq \|x \pm y\| \leq \|x\| + \|y\|$.

Def. Convergence. $\|x\| \longrightarrow \text{dist} \longrightarrow \text{Topology}$.

$x_n \to x$ iff $\|x_n - x\| \to 0$.

closed. open

compact

Equivalence of norm: $\exists\, C > 0$.

$$\frac{1}{C} \|\cdot\|_2 \leq \|\cdot\|_1 \leq C \|\ \|_2$$

Equivalent norms induce the same topology.

Examples.

1) Product Space : $X_1 \times X_2$ with ① $\|(x_1, x_2)\| = \|x_1\| + \|x_2\|$.

② $\|(x_1, x_2)\| = \max\{\|x_1\|, \|x_2\|\}$ ③ $\|(x_1, x_2)\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$

All ①–③ norms are equivalent.

2) $Y \subseteq X$. linear subspace. Def. $\bar{Y}$ = closure of Y under the topology induced by $\|\cdot\|$:

$\forall x \in \bar{Y}$, $\exists \{y_n\} \in Y$. $y_n \to x$ as $n \to \infty$.

Claim: $\bar{Y}$ is a subspace.

Note: NOT all subspaces are closed, in infinite dimensional case.

4) $Y \subseteq X$. closed linear subspace.

$X \setminus Y := \{ [x] \mid x \in X \}$, where $[x] = \{ z \in X \mid x - z \in Y \}$

Def. quotient norm

$$\|[x]\| = \inf \{ \|z\| \mid z \in [x] \}$$

Check: $[0] = Y$.

1) $\|[x]\| \geq 0$ ✓.

2) if $\|[x]\| = 0$: $\exists \{z_n\} \in [x]$. $\|z_n\| \to 0$. $z_n \to 0$

$\Rightarrow$ $z_n - x \to -x \in Y$ $\boxed{\text{since } Y \text{ is closed}}$ $\Rightarrow x \in Y$.

$\iff [x] = [0]$.

3) $\|\alpha[x]\| = \|[\alpha x]\| = \inf\{\|z\| \mid z \in [\alpha x]\}$

$= \inf\{\|z\| \mid z - \alpha x \in Y\}$

If $\alpha \neq 0$: $\frac{1}{\alpha} z - x \in Y$

$\Rightarrow$ $= \inf\{\|z\| \mid \frac{z}{\alpha} \in [x]\}$

$= \alpha \inf\{\|\underbrace{\tfrac{z}{\alpha}}_{:=y}\| \ \underbrace{\tfrac{z}{\alpha}}_{y} \in [x]\}$

$= \alpha \|[x]\|$

if $\alpha = 0$: $\alpha[x] = 0$, $0 \cdot \|[x]\| = 0 = \|[\alpha x]\|$.

4) Triangle ineq:

$\|[x]+[y]\| = \|[x+y]\| = \inf\{\|z\| \mid z \in [x+y]\}$.

$= \inf\{\|z\| \mid z \in [x]+[y]\}$.

Write $z = z_1 + z_2$. where $z_1 \in [x]$. $z_2 \in [y]$.

$= \inf\{\|z_1 + z_2\| \mid z_1 \in [x], z_2 \in [y]\}$

$\leq \inf\{\|z_1\| + \|z_2\| \mid z_1 \in [x], z_2 \in [y]\}$

$\inf(A+B) = \inf A + \inf B$. $= \inf\{\|z_1\| \mid z_1 \in [x]\} + \inf\{\|z_2\| \mid z_2 \in [y]\}$

$= \|[x]\| + \|[y]\|.$ $\square$

Completion of a normed Space.

Def. (Banach) A normed Space X is called Banach Space if X is complete (i.e. all Cauchy Sequences converge to some point in X).

Theorem 1. A normed vector space X admits a (unique) completion.

Proof. Define the space of sequences of X:

$$Z := \{ (x_n)_{n=1}^{\infty} \mid x_n \in X \text{ is Cauchy} \}.$$

$$X_0 = \{ (x, x, x, \cdots) \mid x \in X \}$$

We could indentify X with X_0, hence

X is embedded into Z.

For $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty} \in Z$.

Define $x \sim y$ if $\|x_n - y_n\| \to 0$ as $n \to \infty$.

$Y := \{ (y_n)_{n=1}^{\infty} \mid y_n \in X. \quad y_n \to 0 \}.$

$\Leftrightarrow x \sim y$ iff $x - y \in Y$

Consider the quotient space $\bar{X} := Z \setminus Y$.

Define a norm on $\bar{X}$:

$$\| [x_j] \| = \lim_{j \to \infty} \| x_j \|.$$

Claim 1: $\| [x_j] \|$ defines a norm.

① The limit defined above exists:

Since $(x_j)_{j=1}^{\infty}$ is Cauchy, $\exists k_0$. $k, j > k_0$ $\|x_j - x_k\| < \varepsilon$

$$\big| \|x_j\| - \|x_k\| \big| < \|x_j - x_k\| < \varepsilon$$

$\to \left(\|x_j\| \right)_{j=1}^{\infty}$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$.

$\Rightarrow \lim \|x_j\|$ exists for $(x_j)_{j=1}^{\infty} \in Z$.

② $\| [x_j] \|$ is independent of the choice of representative:

Take $(x_j), (y_j) \in [x_j]$ $\quad (x_j - y_j)_{j=1}^{\infty} \in Y$

$\Rightarrow \|x_j - y_j\| \to 0 \Rightarrow \big| \|x_j\| - \|y_j\| \big| \le \|x_j - y_j\| \to 0$

$\Rightarrow \lim \|x_j\| = \lim \|y_j\|.$

③ $\|[x_j]\|$ is indeed a norm.

(i) : $\|[x_j]\| \geq 0$ trivial since $\|x_j\| \geq 0$

(ii) : If $\|[x_j]\| = 0$: $\lim \|x_j\| = 0 = \lim \|x_j - 0\|$.

$\Leftrightarrow \quad x_j \to 0. \qquad (x_j)_{j=1}^{\infty} \in Y$

$\Leftrightarrow \quad [x_j] = [0].$

(iii) $\|\alpha[x_j]\| = \|[\alpha x_j]\| = \lim \|\alpha x_j\|$

$= |\alpha| \lim \|x_j\|$

$= |\alpha| \, \|[x_j]\|.$

(iv) Triangle inequality:

$\|[x_j] + [y_j]\| = \|[x_j + y_j]\|$

$= \lim \|x_j + y_j\| \leq \lim \|x_j\| + \|y_j\|$

Cauchy $\Rightarrow \|x_n\|$ is bounded, limit is always finite $\longrightarrow$ $= \lim \|x_j\| + \lim \|y_j\|$

$= \|[x_j]\| + \|[y_j]\|.$

Claim 2: $\bar{X}$ is the completion of X wrt the normed defined above, identifying X with $X_0 \subseteq Z$.

① $\bar{X}$ is complete: Take a sequence of elements in $\bar{X}$: $\{[x_j^n]\}_{n=1}^{\infty}$ that is Cauchy in $\|\cdot\|_{\bar{X}}$.

i.e. $\forall \varepsilon > 0$. $\exists N$. $m, n > N$.

$$\|[x_j^m] - [x_j^n]\| < \varepsilon.$$

$$\Leftrightarrow \left| \lim_j \|x_j^m\| - \lim_j \|x_j^n\| \right| < \varepsilon$$

Take $p = 1, 2, \cdots$, $\varepsilon = \frac{1}{2^{p+1}}$, $\exists N = N_p$,

if $m, n \geq N_p \Rightarrow \|[x_j^n] - [x_j^m]\| \leq \varepsilon = \frac{1}{2^{p+1}}$

In particular, $\|[x_j^{N_p}] - [x_j^m]\| \leq \frac{1}{2^{p+1}}$

WLOG, we could assume $N_p < N_{p+1} \cdots$ increasing

$\downarrow R(p)$ $\quad \downarrow S(p) = T(p)$

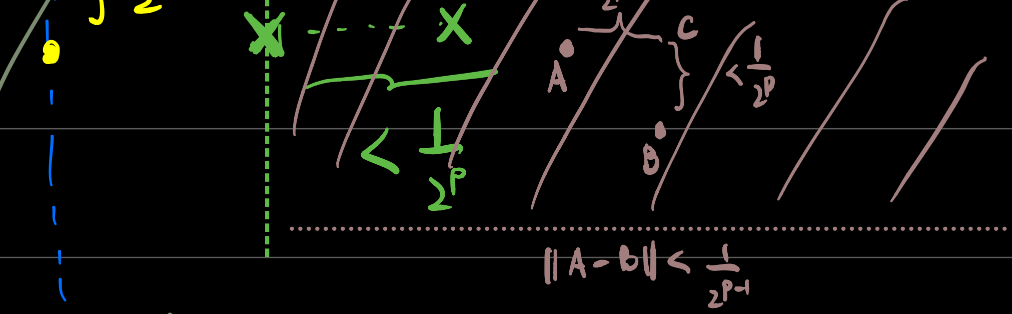
1 $[x_j^1]$: x_1^1, x_1^2, ...

2

N_p $[x_j^{N_p}]$:

N_{p+1} $\quad \Big\} \frac{1}{2^p}$ $\quad < \frac{1}{2^p}$

$[x_j^{N_{p+1}}]$:



$$= \lim_{j\to\infty} \|x_j^{N_p} - x_j^m\| \le \frac{1}{2^{p+1}} \quad \forall m \ge N_p$$

For each $N_p < m \le N_{p+1}$, $\exists R = R(m, N_p)$, $\forall j > R$:

$$\|x_j^{N_p} - x_j^m\| < \frac{1}{2^{p+1}}$$

Set $R = R(p) := \max\limits_{N_p < m \le N_{p+1}} R(m,p)$

we are able to compare elements in each column.

Now compare elements in each row:

Fix $N_p < m \le N_{p+1}$. $(x_j^m)_{j=1}^\infty$ is Cauchy.

$\Rightarrow \exists S(m,p)\ \forall j, k > S(m,p)$

$$\|x_j^m - x_k^m\| < \frac{1}{2^p}$$

$S(p) := \max\limits_{N_p < m \le N_{p+1}} S(m,p)$

$T(p) := \max\{S(p), R(p)\}$.

WLOG. Assume $T(p)$ is increasing strictly.

Conclusion: $\forall\, p>0$. In the region $(N_p, N_{p+1}] \times [T_p, \infty)$

$\| X_k^n - X_j^m \| \leq \frac{1}{2^{p-1}}$ (*)

Construct a sequence $(x_n)_{n=1}^{\infty}$ as: $x_p := X_{T(p)}^{N_p}$

① $(x_p)_{p=1}^{\infty}$ is Cauchy in X:

$$\| x_p - x_{p+1} \| = \| X_{T(p)}^{N_p} - X_{T_{p+1}}^{N_{p+1}} \| \leq \frac{1}{2^{p-1}} \text{ by (*)}$$

which is summable. Hence $(x_p)_{p=1}^{\infty}$ is Cauchy in X.

② $[X_j^m] \xrightarrow{m} [x_j]$:

$$\lim_{m\to\infty} \| [X_j^m] - [x_j] \| \leq \lim_{m\to\infty} \lim \| X_j^m - x_j \|$$

$$= \lim_{m\to\infty} \lim_{j\to\infty} \| X_j^m - X_{T(j)}^{N(j)} \|$$

Say $N(p) < m \leq N(p+1)$. If j is large: Say $j > T(p)$:

$$\| X_j^m - X_{T(j)}^{N(j)} \| \leq \| X_j^m - X_{T(p)}^{N(p)} \| + \| X_{T(p)}^{N(p)} - X_{T(j)}^{N(j)} \|$$

$$\leq \frac{1}{2^{p-1}} + \frac{1}{2^{p-1}} = \frac{1}{2^{p-2}} \text{ by (*)}$$

$$\Rightarrow \lim_{j} \| x_j^m - x_{T(j)}^{N(j)} \| \le \frac{1}{2^{p-2}}$$

Now sending $m \to \infty$, $p \to \infty$ as well since $N(p) < m \le N(p+1)$.

$$\Rightarrow \lim_{m} \lim_{j} \| \quad \| = 0.$$

Hence $\overline{X}$ is complete.

Summary: X Normed.

$Z := \{ (x_n)_{n=1}^{\infty} \mid x_n \in X,\ (x_n) \text{ Cauchy} \}$

$X \approx X_0 := \{ (x, x, x, \cdots) \mid x \in X \}$

$Y := \{ (x_j)_{j=1}^{\infty} \mid x_j \to 0 \}$.

$\overline{X} = Z \setminus Y$.

For $x \in X$, identify with $(x, x, x \cdots) \in X_0$

$\| [x] \| = \lim_{n \to \infty} \| x \| = \| x \|$

$\Rightarrow X$ is embedded in Z, preserving the norm.

② Claim: $\overline{X_0 \setminus Y} = \bar{X}$

Proof of the claim: $\subseteq$ is trivial.

$\supseteq$: take $[x_j] \in \bar{X} = Z \setminus Y$.

Construct a sequence $([x^n])_{n=1}^{\infty}$ in $X_0 \setminus Y$ s.t. $[x^n] \to [x_j]$:

if $(x_1, x_2, \dots) \in [x]$: Let

$$(x^1) = (x_1, x_1, x_1, \dots)$$
$$(x^2) = (x_2, x_2, x_2, \dots)$$
$$\vdots$$
$$(x^p) = (x_p, x_p, \dots)$$

$(x^p) \in X_0$ since it's constant.

And $[x^n] \to [x]$:

$$\|[x^n] - [x]\| = \lim_{n\to\infty} \lim_{j} \| x^n_j \; x_j \|$$

$= x^n = x_n$ by construction.

$$= \lim_{n\to\infty} \lim_{j\to\infty} \| x_n - x_j \|$$

$$= 0 \text{ since } (x_n) \in Z \text{ is Cauchy.}$$