Vector Space.

Throughout we denote $(X, \mathbb{F})$ a vector space over field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

$Y \subseteq X$ is a subspace if

$$\forall y_1, y_2 \in Y, \quad a \in \mathbb{F} \Rightarrow \begin{array}{l} y_1 + y_2 \in Y \\ a y \in Y \end{array}$$

Prop. 1 (i) $\underline{Y_1 + Y_2 := \{ y_1 + y_2 \mid y_i \in Y_i \}}$ is a subspace if both Y_1, Y_2 are.

(ii) $\{ Y_\theta \mid \theta \in I \}$ is a set of subspaces, then

$$\bigcap_{\theta \in I} Y_\theta \text{ is also a subspace.}$$

Proof of (ii): Take $y_1, y_2 \in \bigcap_{\theta \in I} Y_\theta$, then

$$y_1, y_2 \in Y_\theta \quad \forall \theta \in I$$

$$\Rightarrow \quad y_1 + y_2 \in Y_\theta \quad \forall \theta \in I \text{ since } Y_\theta \text{ is a subspace}$$

$$\Rightarrow \quad y_1 + y_2 \in \bigcap_{\theta \in I} Y_\theta$$

The case of ay where $a \in \mathbb{F}$, $y \in \bigcap_{\theta \in I} Y_\theta$ is similar.

$\blacksquare$

Def. $\{Y_\theta\}_{\theta\in I}$ is totally ordered if $\forall \theta_1, \theta_2 \in I$, either $Y_{\theta_1} \subseteq Y_{\theta_2}$ or $Y_{\theta_1} \supseteq Y_{\theta_2}$

Thm 2. If $\{Y_\theta \mid \theta\in I\}$ totally ordered linear subspaces. Then $Y := \bigcup_{\theta\in I} Y_\theta$ is a subspace.

RMK. Counter example: $Y_1 = \{(0,y) \mid y\in\mathbb{R}\} \subseteq \mathbb{R}^2$

$Y_2 = \{(x,0) \mid x\in\mathbb{R}\} \subseteq \mathbb{R}^2$

$Y_1 \cup Y_2$ is NOT a subspace.

Proof: exercise.

Linear Span.

Take $S \subseteq X$ a subset (may not be subspace)

$\mathrm{Span}(S) := \bigcap_{\theta\in I} Y_\theta$ where Y_θ is a subspace of X that contains S.

Prop 3. $\mathrm{Span}(S)$ is the smallest linear subspace containing S.

Proof: Span(S) is a subspace by previous prop. 1 (ii).
Now assume $Y \supseteq S$ is a subspace.
Then $Y = Y_\theta$ for some θ.
$\Rightarrow Y \supseteq \text{Span}(S)$. ∎

Prop 4. $\text{Span}(S) = \left\{ \sum_{j=1}^{n} \alpha_j x_j \;\middle|\; x_j \in S,\ \alpha_j \in \mathbb{F} \right\}$

Proof: Denote $Z := \{ \qquad \}$.
First, Z is a subspace (trivially true).
Second, Z contains S (also trivial)
Last, If Y is a linear subspace containing S, then $Y \supseteq Z$: Take $z \in Z$. $z = \sum_{j=1}^{n} \alpha_j x_j$ where $\alpha_j \in \mathbb{F}$, $x_j \in S$. Then $z \in Y$ because $Y \supseteq S$ and Y is a subspace.
$\Rightarrow Z = \text{Span}(S)$. ∎

Quotient Space.

Given $Y \subseteq X$ a subspace. Define $x \sim y$ iff

$x - y \in Y$. $\forall x, y \in X$.

Then $[x] := \{ y \in X \mid y \sim x \}$.

$X/Y := \{ [x] \mid x \in X \}$. is the quotient space.

X/Y is a vector space if we define

$$[x] + [y] := [x+y]$$

$$\alpha [x] := [\alpha x].$$

RMK. Such operations are well-defined:

If $x_1, x_2 \in [x]$. $y_1, y_2 \in [y]$.

WANT: $[x_1 + y_1] = [x_2 + y_2]$

$\Rightarrow$ $\begin{matrix} x_1 - x \in Y \\ x_2 - x \in Y \end{matrix}$ $\Rightarrow$ $x_1 - x_2 \in Y$ also $y_1 - y_2 \in Y$

Take $z \in [x_1 + y_1]$. $\Rightarrow$ $(x_1 + y_1) - z \in Y$

$$\Rightarrow (x_2 + y_2) - z = \underbrace{(x_2 - x_1)}_{\in Y} + \underbrace{(y_2 - y_1)}_{\in Y} + \underbrace{(x_1 + y_1) - z}_{\in Y}$$

$\in Y$

$\Rightarrow x_2 + y_2 \sim z$ as well. $\Rightarrow [x_1+y_1] \subseteq [x_2+y_2]$

By symmetry, $[x_2+y_2] \subseteq [x_1+y_1]$.

$\Rightarrow \quad [x_2+y_2] = [x_1+y_1]$.

The case of scalar multiplication is similar. ■

Linear maps

X, Y are both vector spaces over $\mathbb{F}$.

Def (i) $T: X \to Y$ is linear if

$$T(x_1+x_2) = T(x) + T(x_2)$$

$$T(\alpha x_1) = \alpha T(x_1)$$

$\forall x_1, x_2 \in X$, $\alpha \in \mathbb{F}$.

(ii) X, Y are isomorphic if $\exists$ bijective linear map between X and Y

Prop 5. (i) $\tilde{X} \subseteq X$ is a subspace of X. Then

$T(\tilde{X}) \subseteq Y$ is a subspace of Y.

(ii) $\tilde{Y} \subseteq Y$ is a subspace of Y.

Then $T^{-1}(\tilde{Y})$ is a subspace of X.

Proof: exercise.

Convexity

Assume $\mathbb{F} = \mathbb{R}$.

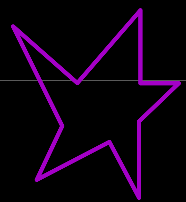
Def. $K \subseteq X$ is **convex** if $\forall x, y \in K$, $\alpha \in [0,1]$

$$\alpha x + (1-\alpha) y \in K.$$

Examples

1) Linear subspaces 2) unit ball. 3) Square

Counter-example:



Prop 6. If K is convex. $x_1, \cdots, x_n \in K$

$\alpha_1, \cdots, \alpha_n \in [0,1]$ s.t $\sum_{i=1}^{n} \alpha_i = 1$. then

the convex combination $\sum_{i=1}^{n} \alpha_i x_i \in K$

Proof. By induction. $n=2$ is trivial.

Assume $n=N$ is true.

$$\sum_{i=1}^{N+1} \alpha_i x_i = \sum_{j=1}^{N} \alpha_j x_j + \alpha_{N+1} x_{N+1}$$

$$\xrightarrow{\text{if } \alpha_{N+1} \neq 1} = (1-\alpha_{N+1}) \sum_{j=1}^{N} \frac{\alpha_j x_j}{1-\alpha_{N+1}} + \alpha_{N+1} x_{N+1}$$

Now $\sum \frac{\alpha_j x_j}{1-\alpha_{N+1}} \in K$ since $\sum_{j=1}^{N} \frac{\alpha_j}{1-\alpha_{N+1}} = 1$.

Call $x := \sum \frac{\alpha_j x_j}{1-\alpha_{N+1}} \in K$.

$(1-\alpha_{N+1}) x + \alpha_{N+1} x_{N+1} \in K$ by convexity.

if $\alpha_{N+1} = 1$: trivial. ■

Prop 7.(i) Given K_1, K_2 convex then

$K_1 + K_2 := \{ x_1 + x_2 \mid x_i \in K_i \}$ is convex.

Proof. Take $z_1, z_2 \in K_1 + K_2$.

Write $\begin{cases} z_1 = x_1 + y_1 \\ z_2 = x_2 + y_2 \end{cases}$ where $x_i \in K_1$, $y_i \in K_2$

$\Rightarrow \forall \alpha \in [0,1]$: $\alpha z_1 + (1-\alpha) z_2 = \alpha(x_1+y_1) + (1-\alpha)(x_2+y_2)$

$$= \underbrace{\alpha x_1 + (1-\alpha) x_2}_{\in K_1} + \underbrace{\alpha y_1 + (1-\alpha) y_2}_{\in K_2}$$

By convexity:

$$\Rightarrow \alpha z_1 + (1-\alpha) z_2 \in K_1 + K_2 \quad \blacksquare$$

7 (ii). Given $\{K_\theta \mid \theta \in I\}$ convex sets.

$$\bigcap_{\theta \in I} K_\theta \text{ is convex}$$

Proof: Similar to Prop 1 (ii) $\blacksquare$

(iii) Given $\{K_\theta \mid \theta \in I\}$ totally ordered convex sets.

Then $\displaystyle\bigcup_{\theta \in I} K_\theta$ is convex.

Proof: Similar to Theorem 2. $\blacksquare$

(iv). K convex. T linear map. Then

$T(K)$ is convex.

Proof: exercise.

(v). K convex. T linear map. Then

$T^{-1}(K)$ is convex.

Proof: exercise.

Convex Hull.

Let $S \subseteq X$ be a subset. Define

$$Co(S) := \bigcap_{\theta} k_\theta \quad \text{where } S \subseteq k_\theta,$$

k_θ is convex.

Prop 8. $Co(S)$ is the smallest convex set containing S.

Proof: Similar to Prop 3.

Prop 9. $Con(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \,\middle|\, x_i \in S,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}$

Proof: Similar to Prop 4.