Finite Dimensional vector Spaces

—— Functional analysis

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Let X be a normed linear space over field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition: $\{z_1, \cdots, z_n\}$ is linearly independent if

$$\sum_{j=1}^{n} \alpha_j z_j = 0 \Rightarrow \alpha_j \equiv 0 \quad \forall j$$

RMK. If $x = \sum_{j=1}^{n} \alpha_j z_j$, the coefficients are unique.

Definition: Let $\{z_1, \cdots, z_N\}$ be a linearly indep. set in X. It's called maximal if

$$\mathrm{Span}(z_1, \cdots, z_N) = X.$$

i.e. $\forall x \in X$, $\exists \alpha_1, \cdots, \alpha_N$

$$x = \sum_{n=1}^{N} \alpha_n z_n$$

Theorem 1 If $\{z_1, \cdots, z_N\}$ and $\{u_1, \cdots, u_M\}$ are both maximal, then

$$N = M.$$

Proof of Thm 1.

Write $u_i = \sum_{j=1}^{N} \alpha_{ij} z_j$

$$z_j = \sum_{l=1}^{M} \beta_{jl} u_l$$

$$\Rightarrow \quad u_i = \sum_{\substack{1 \le j \le N \\ 1 \le l \le M}} \alpha_{ij} \beta_{jl} u_l$$

$$\Rightarrow \sum_{j=1}^{N} \alpha_{ij} \beta_{jl} = \delta_{il}$$

$$(\alpha_{ij})(\beta_{jl}) = Id$$

$$\Rightarrow \quad M = N.$$

$\square$

Definition (Dimension)

The dimension of X is the # of elements in maximal linearly indep set. Call such a set "basis" of X.

RMK. $\dim X = +\infty$ if $\forall k \geq 1$ $\exists$ linearly independent set $\{x_1, \cdots, x_k\} \subseteq X$.

Assume X is a normed space of dimension N. Let $\{z_1, \cdots, z_N\}$ be a basis of X. Then

$$x = \sum_{i=1}^{N} \alpha_i(x)\, z_i \, , \quad \alpha_j \in \mathbb{C}.$$

with unique coefficients $\{\alpha_1, \cdots, \alpha_N\}$.

Lemma 1	$\exists\ C_0 \in \mathbb{R}$. $\left\| \sum_i \alpha_i(x) \right\| \leq C_0 \|x\|$.
Proof.	WLOG. Assume $\|x\| = 1$. Now apply compactness of the unit sphere.
Alternative proof:	Proof by contradiction. Assume such a C_0 does not exist. $\exists\ (x_p)_{p=1}^{\infty} \ \left\| \sum_{i=1}^{N} \alpha_i(x_p) \right\| \geq p \ \|x_p\|$. (*) $y_p := \dfrac{x_p}{\sum_{i=1}^{N} \|\alpha_i(x_p)\|}$

Then $\alpha_j(y_p) = \dfrac{\alpha_i(x_p)}{\sum_{i=1}^{N} \alpha_i(x_p)}$

$\Rightarrow \sum_{j=1}^{N} |\alpha_j(y_p)| = 1$

(*) $\Leftrightarrow \quad 1 \geq p \, \| y_p \|$

$\Leftrightarrow \quad \| y_p \| \leq \frac{1}{p}$.

Consider the sequence of

$\{\alpha_1(y_p)\}_{p=1}^{\infty} : |\alpha_1(y_p)| \leq 1$

By Balzano-Wierestrass of $\mathbb{R}$ or $\mathbb{C}$,

$\exists$ converging subsequence

$\{\alpha_1(y_{p_i})\}_{i=1}^{\infty}$

Repeat the argument for

$\{\alpha_2(y_{p_i})\}_{i=1}^{\infty}$, taking a sub-sub sequence

s.t. $\alpha_2(y_{p_{i_j}})$ converges as well.

Finally, repeating for all coordinates,

we could find a subsequence

$\{p_\ell\}_{\ell=1}^{\infty}$ s.t.

$$\alpha_j(y_{p_\ell}) \to \alpha_j \in \mathbb{C} \quad \forall\ 1 \le j \le N.$$

$$\Rightarrow y_{p_\ell} = \sum_j \alpha_j(y_{p_\ell}) z_j$$

Letting $\ell \to \infty$. $|\sum \alpha_j| = 1$

However, $\|y_{p_\ell}\| \le \frac{1}{p_\ell} \to 0$

$$\Updownarrow$$

$$\alpha_j(y_{p_\ell}) \to 0.$$

Contradiction!

$\square$

Theorem 3. $\overline{B(0,1)} = \{x \in X \mid \|x\| \le 1\}$ is compact.

Proof of Theorem 3. Consider a sequence of points in $\overline{B(0,1)}$: $\{x_p\}_{p=1}^{\infty}$

By previous Lemma,

$$\sum_{j=1}^{N} |\alpha_j(x_p)| \le C_0 \|x_p\| = C_0$$

$\Rightarrow$ Fix j, $|\alpha_j(x_p)| \leq C_0 \quad \forall p$.

$\Rightarrow \exists$ subsequence p_ℓ s.t.

$$\alpha_j(p_\ell) \longrightarrow \alpha_j \quad \text{as } \ell \to \infty.$$

$$\Rightarrow \quad x_{p_\ell} \longrightarrow \sum_{j=1}^{N} \alpha_j z_j$$

Call $y := \sum_{j=1}^{N} \alpha_j z_j$

$$\|y\| \leq \underbrace{\|y - x_{p_\ell}\|}_{\to 0} + \|x_{p_\ell}\|$$

$$\leq \quad \varepsilon \to 0 + 1 \quad \Rightarrow y \in B(0,1).$$

Sequential compactness is shown.

$\square$

Theorom 4 | In finite dimensional vector space, all norms are equivalent.

Proof of Theorem 4:

Define a norm $\|\cdot\|_0$ on X:

$$\|x\|_0 = \sum_{i=1}^{N} |\alpha_i(x)|$$

Exercise: $\|\cdot\|_0$ is indeed a norm.

Claim: $\|\cdot\|$ is equivalent to $\|\cdot\|_0$.

① $\|x\|_0 \le C_0 \|x\|$ as previous lemma shows.

② $\|x\| \le C \|x\|_0$:

$$\|x\| = \left\| \sum_{j=1}^{N} \alpha_j(x) z_j \right\|$$

$$\leq \sum_{j=1}^{N} |\alpha_j(x)| \, \|z_j\|$$

$$\leq \underbrace{\max\{\|z_j\| \mid 1 \leq j \leq N\}}_{C} \cdot \|x\|_0$$

$\square$

Theorem 5 Finite dimensional vector spaces are complete.

proof of Theorem 5: Given a Cauchy sequence $(x_n)_{n=1}^{\infty} \in X$.

$\forall \varepsilon > 0$. $\exists n_0$. $p, q \geq n_0$

$$\|x_p - x_q\| \leq \varepsilon.$$

But since norm are equivalent,

$$\|x_p - x_q\|_0 \lesssim \varepsilon$$

up to a constant

$$\Longleftrightarrow \sum_{j=1}^{N} |\alpha_j(x_p) - \alpha_j(x_q)| \lesssim \varepsilon$$

$\Longleftrightarrow$ Fix j. $|\alpha_j(x_p) - \alpha_j(x_q)| \lesssim \varepsilon$

$\{\alpha_j(x_p)\}_{p=1}^{\infty}$ is Cauchy in $\mathbb{C}$

$$\alpha_j := \lim_{p\to\infty} \alpha_j(x_p)$$

$$x := \sum_{j=1}^{N} \alpha_j z_j$$

$$\|x_p - x\|_0 = \|\sum \alpha_j(x_p) - \sum \alpha_j\|$$

$$\le \sum_{j=1}^{N} \|\alpha_j(x_p) - \alpha_j\|$$

$$\longrightarrow 0.$$

$\Rightarrow x_p \longrightarrow x$ $\square$