Infinite dimensional Vector Space

—— Functional Analysis

XIA Yusen
夏雨森

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We prove a significant property of infinite dimensional vector Space.

Theorem 1 Let X be an infinite dimensional Vector Space. Then the unit ball

$$B(0,1) := \{ x \in X \mid \|x\| \le 1 \}$$

is NOT compact in X.

Remark Actually this is an "iff" statement:

$$B(0,1) \text{ non compact}$$
$$\Updownarrow$$
$$X \text{ is of infinite dimension.}$$

We first prove a lemma:

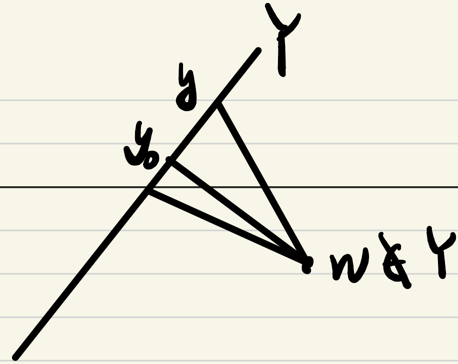
Lemma 2. $Y \subseteq X$ closed subspace.

$$X \setminus Y \neq \emptyset.$$

$$\Rightarrow \forall \varepsilon > 0, \exists z \in X \quad \|z\| = 1, \quad \|z - y\| \ge 1 - \varepsilon \quad \forall y \in Y.$$

Proof of the Lemma:

Fix $\varepsilon > 0$.



$d := \inf \{ \|w-y\| \mid y \in Y, w \notin Y \}$.

Claim: $d > 0$.

Assume $d = 0$. $\exists\, y_n \in Y$.

$\|w - y_n\| \longrightarrow 0$.

$\Longleftrightarrow y_n \to w \Longleftrightarrow w \in Y$

since Y is closed.

① Let's assume $\exists\, y_0$, $\|w - y_0\| = d$.

$$z := \frac{w - y_0}{\|w - y_0\|}$$

then $\|z\| = 1$.

Take $y \in Y$. $\|z - y\| = \left\| \frac{w - y_0}{\|w - y_0\|} - y \right\|$

$$= \frac{1}{\|w-y_0\|} \| w - \underbrace{y_0 - y\|w-y_0\|}_{\in Y} \|$$

$$\geq \frac{d}{d} = 1$$

② if y_0 does not exist:

Let $\delta > 0$ $\exists\, y_0 \in Y$.

$$d \leq \|w-y_0\| \leq (1+\delta)\, d$$

since $d = \inf\{- \cdots\}$

$$z := \frac{w-y_0}{\|w-y_0\|} \text{ has norm } 1$$

$\forall\, y \in Y$: $$\|z-y\| = \frac{1}{\|w-y_0\|} \| w - \underbrace{y_0 - \|w-y_0\| \cdot y}_{\in Y} \|$$

$$\geq \frac{d}{(1+\delta)d} = \frac{1}{1+\delta}$$

Choose δ s.t. $1-\varepsilon = \frac{1}{1+\delta}$

$$\Longleftrightarrow \delta = \frac{\varepsilon}{1-\varepsilon} > 0$$

This completes the prove of the lemma. $\square$

Proof of Theorem 1:

Goal: Construct $\{z_n\}$. $\|z_n\| = 1$

$\|z_n - z_j\| \geq \frac{1}{2}$ $\forall n \neq j$. linearly independent

Choose $\omega \in X$. $\omega \neq 0$.

$z_1 := \frac{\omega}{\|\omega\|}$ has unit norm.

Induction step: Assume $\{z_1, \cdots, z_n\}$ has been chosen s.t.

$\|z_i\| = 1$. $\|z_i - z_j\| \geq \frac{1}{2}$ $\forall i \neq j$.

Construct z_{n+1} as follows:

$Y_n := \operatorname{Span}\{z_1, \cdots, z_n\}$

is a closed subspace of X

$\exists\, \omega \notin Y_n$ since $\dim X = +\infty$.

Apply the previous lemma with $\varepsilon = 1/2$:

$\exists\, z_{n+1} \notin Y_n$. $\|z_{n+1}\| = 1$

$\|z_{n+1} - z_j\| \geq 1 - \varepsilon = \frac{1}{2}$

Show z_{n+1} is linearly independent with $\{z_1, \cdots, z_n\}$:

Assume $\alpha z_{n+1} + \sum_{j=1}^{n} \alpha_j z_j = 0$

if $\alpha = 0$: $\sum \alpha_j z_j = 0 \Rightarrow \alpha_j = 0$ Done.

if $\alpha \neq 0$: $z_{n+1} = -\sum_{j=1}^{n} \frac{\alpha_j}{\alpha} z_j \in \text{Span}(z_1, \cdots, z_n)$

But $z_{n+1} \notin Y_n$ by construction.

Hence the desired sequence $\{z_n\}_{n=1}^{\infty}$ is constructed.

Now show $B(0,1)$ is NOT Compact:

Assume not. Then $\{z_n\}$ has a subsequence

$\{z_{n_k}\} \to x \in X$

$\Rightarrow \{z_{n_k}\}$ is Cauchy Contradiction.

The proof has been completed. $\square$

Definition. (Separable)

A normed vector space is **separable** if $\exists$ countable dense subset.

Example

1) $(\ell^\infty, \| \ \|_\infty)$ is NOT Separable.

2) $M := \{$ signed measures on $[-1,1]$ with Borel sigma algebra $\}$ NOT separable.

Decompose $\mu \in M$ as:

$$\mu = \mu^+ - \mu^-$$

where $\mu^\pm$ are measures.

$$\|\mu\| := \|\mu_+\|_{[-1,1]} + \|\mu_-\|_{[-1,1]}.$$

Then $(M, \| \ \|)$ is a normed vector space.

Let δ_x be the dirac measure $x \in [-1,1]$.

$$\delta_x(A) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

if $x \neq y$: $\|\delta_x - \delta_y\| = 1 + 1 = 2$

Argue by contradiction: assume $\exists$ dense countable set $D \subseteq M$.

Define $F: [-1, 1] \longrightarrow D$ as:

$x \in [-1, 1]$. consider $\delta_x \in M$.

By density, $\exists\ \mu_x \in D$. $\|\mu - \delta_x\| < \frac{1}{2}$

$F(x) := \mu_x$.

Claim: F is injective.

if $\mu_x = \mu_y$:

if $x \neq y$: $\|\delta_x - \delta_y\| \leq \underbrace{\|\delta_x - \mu_x\| + \|\delta_y - \mu_x\|}_{< 1}$

impossible. Hence $\delta_x = \delta_y$.

Hence $F: [-1, 1] \to F([-1, 1]) \subseteq D$

is bijective.

$\Rightarrow$ $\mathbb{D}$ is not countable since $[-1,1]$ is NOT.