INTEGRAL THEORY

Simple functions.

$s(x) = \sum_{j=1}^{\boxed{N}} \alpha_j \mathbb{1}_{A_j}$, where $\alpha_j \in (0,+\infty)$, A_j measurable, is called simple function.

Note. N should be finite.

Thm. (simple function approximation). if f is a positive measurable function, then

$\exists \{\varphi_n\}_{n=1}^{\infty}$ simple functions. $\varphi_n(x) \to f$ pointwise. $\varphi_n(x) \le \varphi_{n+1}(x)$.

Proof: Stein's book.

Integral of Postive measurable functions

For $f: X \to [0,+\infty]$ be measurable function. E be measurable set.

$$\int_E f\,d\mu := \sup_{0 \le s \le f} \int_E s\,d\mu \quad \text{where } s \text{ is simple.}$$

Properties of Postive measurable functions.

(a) Monotone. $0 \le f \le g \Rightarrow \int_E f\,d\mu \le \int_E g\,d\mu$.

(b) if $A \subseteq B$. $\int_A f\,d\mu \le \int_B f\,d\mu$.

(c) $C \ge 0$. $\int_E Cf\,d\mu = c\int_E f\,d\mu$.

(d) $f \equiv 0$. a.e. $x \in E$, then $\int_E f\,d\mu = 0$ even if $\mu(E) = +\infty$

(e) if $\mu(E) = 0$, then $\int_E f\,d\mu = 0$ even if $f(x) = +\infty$.

convention: $0 \cdot \infty = 0$.

(f) $\int_E f\,d\mu = \int f \mathbb{1}_E\,d\mu$.

Theorem. Lebesgue Monotone Convergence Theorem.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions satisfying:

$$0 \le f_1(x) \le f_2(x) \le \cdots$$ Monotone increasing, positive.

Define $f := \lim_{n\to\infty} f_n(x)$, $\forall x \in X$. then we have.

(a) f is measurable. (b) $\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E \lim_{n\to\infty} f_n\,d\mu$, $\forall$ measurable E.

trivially follows from measurable functions

Proof of (b): " $\le$ ": $f_n \le f$. So $\int_E f_n \, d\mu \le \int_E f \, d\mu$. $\forall n$. taking limit yeids $\le$.

" $\ge$ ": key: leave yourself an ε of room. $\lim_{n\to\infty} \int f_n \, d\mu \ge \int f \, d\mu \iff \lim_{n\to\infty} \int_X f_n \, d\mu > \varepsilon \int_X f \, d\mu$, $\forall \varepsilon \in (0,1)$

Let $s(x)$ be simple. $\int f \, d\mu = \sup_{s \le f} \int s \, d\mu$.

Take $\varepsilon \in (0,1)$, $s(x) \le f(x) \Rightarrow \varepsilon s(x) < f(x)$. $\forall x$ s.t. $f(x) \ne 0$. allowed $+\infty$, nonzero.

Let $E_n = \{ x \in X \mid f_n(x) \ge \varepsilon s(x) \}$. since $f_n \uparrow$. $E_1 \subseteq E_2 \subseteq \cdots$ i.e. $E_n \uparrow$.

So $\int_{E_n} f_n \, d\mu \ge \int_{E_n} \varepsilon s(x) \, d\mu$. $(*)$

Claim: $X = \bigcup_{n=1}^{\infty} E_n$. (Thanks to ε-trick)

Since $f_n \to f(x) \ge \varepsilon s(x)$, for $\forall x \in X$, $\exists N$. large. s.t. $f_N(x) > \varepsilon s(x) \Rightarrow x \in E_N$ for some N. So $X = \bigcup_{n=1}^{\infty} E_n$

Taking $(*)$ limit: $\lim_{n\to\infty} \int_X f_n \, d\mu \overset{\text{since } E_n \subseteq X}{\ge} \lim_{n\to\infty} \int_{E_n} f_n \, d\mu \ge \lim_{n\to\infty} \int_{E_n} \varepsilon s(x) \, d\mu \overset{?}{=} \int_X \varepsilon s(x) \, d\mu$

Taking $\varepsilon \to 1^-$: $\lim_{n\to\infty} \int_X f_n \, d\mu \ge \int_X s(x) \, d\mu$. $\forall s(x) \le f$. So $\lim \int f_n \, d\mu \ge \int f \, d\mu$

Why ? holds: Let s be simple. $E_1 \subseteq E_2 \subset \cdots \uparrow E$. $\lim_{n\to\infty} \int_{E_n} s \, d\mu = \int_X s \, d\mu$.

$s = \sum \alpha_i \mathbb{1}_{A_i}$ by cts of measure (from below): $\mu(A_i \cap E_n) \to \mu(A_i \cap E)$ if $E_n \uparrow E$. So does the integral.

Generalization: if $f_n \ge g$ for some integrable g, still can be done.

Alternative proof of (b): See IMPA Real Analysis Video

Let $\{ s_{n,k} \}_{k=1}^{\infty}$ be a sequence of Simple functions approximate f_n

key idea:

$$\begin{array}{ccccc}
\text{max} & \text{max} & \text{max} & & \\
\boxed{s_{11}} & s_{12} & s_{13} & \cdots \longrightarrow & f_1 \\
s_{21} & s_{22} & s_{23} & \cdots \longrightarrow & f_2 \\
s_{31} & s_{32} & s_{33} & \cdots \longrightarrow & f_3 \\
 & & & & \vdots \\
 & & & & \downarrow \\
 & & & & f
\end{array}$$

$g_1 = s_{11}$, $g_2 = \max\{ s_{12}, s_{22} \}$

$\vdots$

$g_n = \max\{ s_{1n}, s_{2n}, \dots, s_{nn} \}$

Exercise: proof g_n are simple, $\to f$. (Using monotone of f_n)

Corollary. Interchanging $\sum_{n=1}^{\infty}$ and $\int d\mu$.

Suppose $f_n(x): X \to [0,+\infty]$ measurable. $f := \sum_{n=1}^{\infty} f_n(x)$. then f is measurable. and $\sum_{n=1}^{\infty} \int f_n d\mu = \int \sum_{n=1}^{\infty} f_n d\mu$.

Remark. if we take $\mu =$ counting measure on X. then $\{a_{i,j} > 0\}_{i,j}$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Fatou Lemma. $f_n: X \to [0,+\infty]$ measurable. (Without monotone assumption.)

$$\int \underbrace{\liminf_{n\to\infty} f_n(x)}_{= \sup_{k\geq 1}(\inf_{n\geq k} f_n(x))} d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$$

Proof: MCT $\Rightarrow$ Fatou. $g_k = \inf_{n\geq k} f_n(x)$. So $g_k(x)\uparrow$, $\to \sup_{k\geq 1} g_k(x) = \liminf_{n\to\infty} f_n(x)$

By MCT. LHS $= \int \lim_{k\to\infty} g_k d\mu = \lim_{k\to\infty} \int g_k d\mu = \lim_{k\to\infty} \int \inf_{n\geq k} f_n d\mu \leq \lim_{k\to\infty} \int f_j d\mu$, $\forall j \geq k$, take $j = k$.

$= $ RHS.

Generalization: (1) if $f_n \leq 0$, then $\int \limsup_{n\to\infty} f_n d\mu \geq \limsup_{n\to\infty} \int f_n d\mu$ by $\overline{\lim}(f_n) = \underline{\lim}(-f_n)$.

(2) $f_n \geq g$. for some integral g. Fatou holds

(3) $f_n \leq g$ for some g. (a) still holds.

Def. Integration of general Real function. $f = f^+ - f^-$. f is integrable iff.

$\overset{\text{def}}{\iff}$ f^+, f^- both integrable. $\int f^{\pm} d\mu < +\infty$

Def. Integration of complex-valued functions. $f: X \to \mathbb{C}$. $= \underbrace{f_1 + i f_2}_{\text{real-valued}}$.

f measurable $:\overset{\text{def}}{\iff}$ f_1, f_2 are measurable in real sense.

Define $L^1(\mu) = \{$ complex measurable $f \mid \int \underbrace{|f|}_{\text{real-valued function}} d\mu < +\infty \}$, (Absolute integrable space)

$|f| = \sqrt{f_1^2 + f_2^2}$.

$$\int f d\mu = \int f_1 d\mu + i \int f_2 d\mu.$$

Thm. $f \in L^1(\mu, \mathbb{C})$. $\left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu$

*Proof: $z = \int_X f\,d\mu \in \mathbb{C}$, then exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. $\alpha z = |z|$ (Rotation)

$$\underbrace{\left|\int f\,d\mu\right|}_{|z|} = \underbrace{\int \alpha f\,d\mu}_{\alpha z \in \mathbb{R}} = \int \mathrm{Re}(\alpha f)\,d\mu + i\underbrace{\int \mathrm{Im}(\alpha f)\,d\mu}_{=0}$$

$$= \int \mathrm{Re}(\alpha f)\,d\mu$$

$$\le \int |\alpha f|\,d\mu = |\alpha| \int |f|\,d\mu = \int |f|\,d\mu.$$

Thm. Suppose $f: X \to [0, +\infty]$. measurable. Define $\nu(A) = \int_A f\,d\mu$. $\forall A \in \mathcal{A}$.

Then ν is also a measure and $\int g\,d\nu = \int g \cdot f\,d\mu$. $\forall$ measurable $g: X \to [0, +\infty]$.

Remark: write $f = \frac{d\nu}{d\mu}$, $d\nu = f\,d\mu$. We say $\nu \ll \mu$, meaning that $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

The converse is so called Radon-Nikodym Theorem.

* Dominated Convergence Theorem.

If $|f_n| < g$ for some $g \in L^1(\mu)$, $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in X$. then

$$\lim_{n\to\infty} \int |f_n - f|\,d\mu = 0.$$

Remark. The result implies $\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$. $\because \left|\int f_n - \int f\right| = \left|\int (f_n - f)\right| \le \int |f_n - f| \to 0$.

Standard proof: Elegant, but not understanding.

$$|f_n - f| \le |f_n| + |f| \le 2g.$$

$$\int \lim_{n\to\infty} \left[2g - |f_n - f|\right] d\mu = \int \liminf_{n\to\infty} [2g \underbrace{- |f_n - f|}_{\ge 0}]\,d\mu$$

$$\overset{\text{Fatou}}{\le} \liminf_{n\to\infty} \int [2g - |f_n - f|]\,d\mu$$

$$\Rightarrow \int 2g\,d\mu \le \liminf_{n\to\infty} \int [2g - |f_n - f|]\,d\mu$$

$\Rightarrow \limsup_{n\to\infty} \int |f_n - f|\,d\mu \le 0$. but $|f_n - f| \ge 0$. So the equality holds. ∎

Alternative. proof.

Reduction: $\tilde{f}_n = f_n - f$. ∴ W.L.O.G to assume : $\begin{cases} f_n \geq 0. \\ f_n \to 0 \ \forall x \in X. \\ f_n \leq g \in L^1(\mu) \end{cases}$

WANT: $\lim\limits_{n\to\infty} \int f_n \, d\mu = 0.$

ε-trick: pick $\varepsilon > 0$ arbitrary.

$$\int f_n \, d\mu = \int_{\{f_n \geq \varepsilon g\}} f_n \, d\mu + \int_{\{f_n \leq \varepsilon g\}} f_n \, d\mu$$

$$\leq \int_{\{f_n \geq \varepsilon g\}} g \, d\mu + \int_X \varepsilon g \, d\mu \qquad = \varepsilon \big(\underbrace{\textstyle\int_X g \, d\mu}_{\text{finite.}} \big)$$

Using cts of measure to show $\mu(\{f_n \geq \varepsilon g\}) \to 0$.

$A_n = \{ x \in X \mid f_n(x) \geq \varepsilon g(x) \}$

Since $f_n \leq g$, $f_n \to 0$. $\bigcup_{n=1}^{\infty} A_n = X$

(待补充)

Finally, take $\varepsilon \to 0^+$. done.

3rd proof. Fatou ⇒ DCT. $|f_n| \leq g \Rightarrow \underbrace{-g \leq f_n} \underbrace{f_n \leq g}$.

$\Rightarrow \overline{\lim} \int f_n d\mu \leq \int \overline{\lim} \, f_n \, d\mu = \int \lim f_n \, d\mu = \int \underline{\lim} \, f_n d\mu \leq \underline{\lim} \int f_n d\mu$. So everything is equal.

since $f_n \to f$.

a.e. Generalized DCT. Suppose f_k, g_k satisfying.

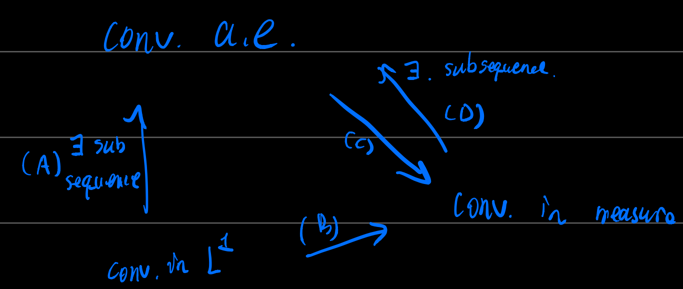
(1) $|f_k| < g_k$. a.e. for each k (Almost dominated)

(2) $\lim f_n = f$ a.e. $\lim g_k = g$ a.e.

(3) $\lim\limits_{k\to\infty} \int_X g_k \, d\mu = \int_X g \, d\mu < +\infty$ ($g \in L^1(\mu)$).

then. $\lim\limits_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu$.

Convergence modes



(A). Assume $\{f_n\}_{n=1}^{\infty}, f \in L^1(\mu)$. $f_n \xrightarrow{L^1} f$.

i.e. $\lim_{n\to\infty} \int |f_n - f| d\mu = 0$

then. $\exists \{f_{n_k}\}_{k=1}^{\infty}$. $f_{n_k} \longrightarrow f$. a.e.

proof: Since $\lim_{n\to\infty} \int |f_n - f| d\mu = 0$, $\exists$ a subsequence $\{f_{n_k}\}$. s.t.

$\int | f_{n_k} - f | d\mu < 2^{-k}$.

$\Rightarrow \sum \int | f_{n_k} - f | d\mu < \sum_{k=1}^{\infty} 2^{-k} = 1 < +\infty$.

MCT.

$\Rightarrow \int \sum | f_{n_k} - f | d\mu = 1 < +\infty$

$\Rightarrow \sum_{k=1}^{\infty} |f_{n_k} - f|$ is finite a.e. (Integrable $\Rightarrow$ Bounded a.e)

$\Rightarrow |f_{n_k}(x) - f(x)| \to 0$ a.e.

Remark: Why "sub" sequence? Counter-example. "Walking stick".

$f_1 = \mathbb{1}_{[0,\frac{1}{2}]}$. $f_2 = \mathbb{1}_{[\frac{1}{2},1]}$

$f_3 = \mathbb{1}_{[0,\frac{1}{4}]}$. $f_4 = \mathbb{1}_{[\frac{1}{4},\frac{1}{2}]}$ $f_5 = \mathbb{1}_{[\frac{1}{2},\frac{3}{4}]}$ $f_6 = \mathbb{1}_{[\frac{3}{4},1]}$. ..... $f_n \xrightarrow{L^1} 0$

each $x \in [0,1]$, $\exists$ infinitely many n. $f_n(x) \neq 0$.

$\therefore$ No where conv to 0. concerning the whole sequence.

Thm. Suppose $\{f_n\}_{n=1}^{\infty}$ complex valued. measurable. defined a.e. on X, with $\sum_{n=1}^{\infty} \int |f_n| d\mu < +\infty$

Then $f := \sum_{n=1}^{\infty} f_n$ conv. a.e. and $f \in L^1(\mu)$.

$\int_X f d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$. Interchanging $\sum_{n=1}^{\infty}$ and $\int$

Analogue to positive function case (proved by MCT).

Proof: Let S_n be the set f_n is defined. $\mu(S_n^c) = 0$.

let $S = \cap S_n$. $\forall f_n$ defined on S. $\mu(S^c) = \mu(\cup S_n^c) = 0$.

Set $\varphi(x) = \sum_{n=1}^{\infty} |f_n(x)|$. $\varphi \in L^1$.

$f_N(x) = \sum_{n=1}^{N} f_n(x)$. $|f_N| \leq \varphi$.

By DCT. $\lim f_N = f$.

Remark: this is a baby version of Fubini Thm.

$\tilde{f}(n,x) := f_n(x)$. Let ν be counting measure on $\mathbb{N}$.

The above Thm $\Leftrightarrow$

$$\sum_{n=1}^{\infty}\int_X |f_n| d\mu = \int_{\mathbb{N}\times X} |\tilde{f}(n,x)| d\mu\, d\nu < +\infty$$

$$\underset{\text{Fubini}}{=} \int_{\mathbb{N}\times X} |\tilde{f}(n,x)| d\nu\, d\mu = \int_X \sum_{n=1}^{\infty} |f_n(x)| d\mu.$$

Thm. (a) Suppose $f: X \to [0,+\infty]$ measurable. $\int_E f d\mu = 0 \quad \forall E \in \mathcal{A}$.

then, $f = 0$ a.e.

Proof: $E_n = \{x \in X \mid f(x) > \frac{1}{n}\}$ $\quad \{f(x) > 0\} = \bigcup E_n$.

$$\mu(E_n) = \int_{E_n} 1 d\mu = \int_{E_n} n \cdot \tfrac{1}{n} d\mu < n \cdot \int_{E_n} f d\mu = 0. \text{ by assumption}$$

$$\Rightarrow \mu(\{f(x) > 0\}) \leq \sum \mu(E_n) = 0.$$

(b). $f \in L^1(\mu)$, complex valued. measurable. and

$$\int_X |f| d\mu \overset{\geq \text{ is automatic}}{=} \left|\int_X f d\mu\right|.$$

then $\exists\, \alpha \in \mathbb{C}$. $\alpha f = |f|$ a.e.

Proof: $z := \int_X f d\mu \in \mathbb{C}$ take $\alpha \in \mathbb{C}$. $|\alpha| = 1$ s.t. $\alpha z = |z| \in \mathbb{R}$.

RHS $= |z| = \alpha z$

$$\text{LHS} - \text{RHS} = \int_X \left[|f| - \alpha f\right] d\mu = 0$$

$\Rightarrow |f| = \alpha f$ a.e (by (a)). ∎

Thm. (Borel Cantelli 1st) Let $\{E_n\}_{n=1}^{\infty} \in (X, \mathcal{F}, \mu)$. if $\sum_{n\geq 1}\mu(E_n) < +\infty$. then

$$\mu(E_n \text{ i.o.}) = \mu\Big(\underbrace{\bigcap_{n=1}^{\infty}\bigcup_{m\geq n} E_m}_{:= \limsup_{n\to\infty} E_n}\Big) = 0.$$

Proof:

$$g(x) = \sum_{k=0}^{\infty} \mathbb{1}_{E_k} \qquad \int_X g(x) d\mu = \sum \mu(E_k) \overset{\text{integrable}}{<+\infty} \Rightarrow \boxed{g(x) < +\infty \text{ a.e.}}$$

$\Rightarrow$ let $E := \{x \in$ infinitely many $E_k\} = \{E_n \text{ i.o.}\}$ has measure zero. ∎

Application of DCT.

Thm. Suppose $f = f(t,x) : [a,b] \times \mathbb{R} \to \mathbb{C}$. Assume for each $t \in [a,b]$, the function $f(t,\cdot)$ is integrable. Define

$$F(t) := \int_X f(t,x)\, d\mu \leftarrow \text{对 } x \text{ 积分}$$

(a) Suppose $\exists\, g(x) \in L^1(\mu)$, $|f(x,t)| \le g(x)$ a.e. x, if $\lim_{t\to t_0} f(t,x) = f(t_0,x)$

then $\lim_{t\to t_0} \int_X f(t,x)\, d\mu = \int_X f(t_0,x)\, d\mu$.

In particular, if $f(t,\cdot)$ is cts, then $F(t)$ is cts.

(b) Suppose $\frac{\partial f}{\partial t}$ exists and $\left|\frac{\partial f}{\partial t}\right|(t,x) \le g(x)$ a.e. for all x, t. Then F is diff. and $F'(t) = \int_X \frac{\partial f}{\partial t}(t,x)\, d\mu$.

Proof. (Idea)

(a) take arbitrary sequence $\{t_n\}_{n=1}^{\infty} \to t_0$. $\lim_{n\to\infty} \int_X \underbrace{f(t_n,x)}_{|f(t_n,x)| \le g(x).}\, d\mu \overset{\text{D.C.T.}}{=} \int_X f(t_0,x)\, d\mu$

(b) Fix $\forall\, t_0 \in [a,b]$. $\frac{\partial f}{\partial t}(t_0,x) = \lim_{t_n\to t_0} \frac{f(t_n,x) - f(t_0,x)}{t_n - t_0}$

$$\int \lim_{t_n\to t_0} \boxed{\frac{f(t_n,x) - f(t_0,x)}{t_n - t_0}} \overset{?}{=} \int_X \frac{\partial f}{\partial t}(t_0,x)\, d\mu.$$

$\frac{\partial f}{\partial t}$ exists and bounded around a nbhd. of t_0.

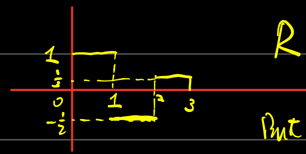
$\le \left|\frac{\partial f}{\partial t}(\tau, x)\right|$ for some $\tau \in (t_n, t_0)$ or (t_0, t_n)

$\le g(x)$.

And then D.C.T. ∎

Lebesgue vs. Riemann.

Confusing "counter-example." $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^k}{n}\mathbb{1}_{[n-1,n)}$



R-integrable since $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots$ conv.

But NOT L-integrable since $\sum\frac{1}{n}$ diverges.

(by def. L-int $\Leftrightarrow$ $|f|$ L-int.)

Thm. Suppose $f:[a,b]\to\mathbb{R}$. bounded.

a) if R-int, then L-int. and $\int_a^b f\,dx=\int_{[a,b]}f\,d\mu$

Proof: f R-int $\to$ measurable $\to$ L int.

Define $G_P(x)=\sum\max_{x\in[x_i,x_{i+1}]}f(x)\,\mathbb{1}_{[x_i,x_{i+1}]}$ $\quad g_P(x)=\sum_i\min_{[x_i,x_{i+1}]}f\,\mathbb{1}_{[x_i,x_{i+1}]}$ for some partition P.

f Rie. $\Rightarrow$ $g_P(x)\le f(x)\le G_P(x)$. Refine partition P. $P_n\subseteq P_{n+1}\cdots$

simple functions

Then by R-int. of $f(x)$, we have $\int\underbrace{\lim_{n\to\infty}G_{P_n}(x)}_{:=G(x)}dx=\int\underbrace{\lim_{n\to\infty}g_{P_n}(x)}_{:=g(x)}dx$

But $G(x)\ge f(x)\ge g(x)$. So $g(x)=f(x)=G(x)$ a.e.

$g_{P_n}(x)\uparrow f$ and $G_{P_n}(x)\downarrow f$. as $n\to\infty$, a.e. $\Rightarrow$ f is L-integrable (Outer int = inner int.)

$\int_{[a,b]}f\,d\mu=\int_a^b f(x)\,dx$ ■

(b) f R-int. iff. f is cts. a.e..

RMK. Characterization of discontinuity. Denote $H(x)=\lim_{\delta\to0}\sup_{|y-x|\le\delta}f(y)$. $\quad h(x)=\lim_{\delta\to0}\inf_{|y-x|\le\delta}f(y)$

f is cts at $x\in\mathbb{R}$ $\Leftrightarrow$ $H(x)=h(x)$ (证明从略)