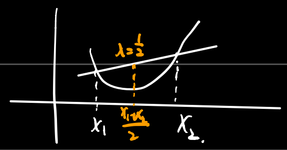
L^p SPACES

Ref. Chapter 6, Folland

1. Convex functions and inequalities

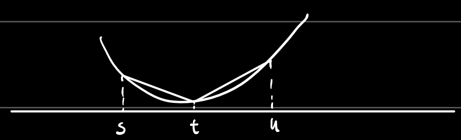
(High School Jensen Inequality)



$f:(a,b) \to \mathbb{R}$ is called "convex" if $f((1-\lambda)x+\lambda y) \le (1-\lambda)f(x)+\lambda f(y)$

$\forall \lambda \in [0,1]$, $x, y \in (a,b)$. (a,b) could be $(-\infty, b)$, $(a,+\infty)$ or $(-\infty,+\infty)$

RMK. Convexity $\Leftrightarrow$ $\dfrac{f(t)-f(s)}{t-s} \le \dfrac{f(u)-f(t)}{u-t}$, $s<t<u$.



$\Leftrightarrow$ "Slope" is $\uparrow$ $\Leftrightarrow$ $f''(x) \ge 0$. for C^2 function $\forall x \in (a,b)$

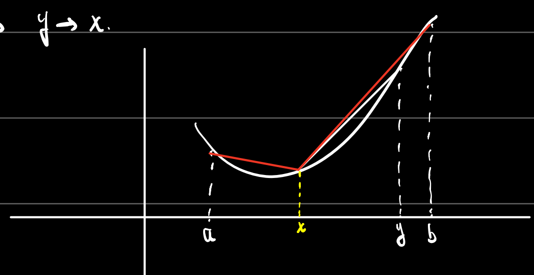
$\Updownarrow$

f' is $\uparrow$ for C^1 function

Thm. f is convex on (a,b) $\Rightarrow$ f is cts.

proof. high school Geometry + calculus 留作习题

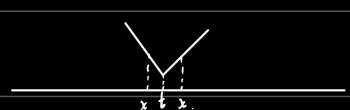
WANT: $f(y) \to f(x)$ as $y \to x$.



$(y, f(y))$ is sandwiched between red lines.

RMK. f convex, fix t. $\lim\limits_{x \to t} \dfrac{f(x)-f(t)}{x-t}$ may not exist, but $\liminf\limits_{x\to t}(\quad) = l_-$, $\limsup\limits_{x\to t}(\quad) = l_+$.

?



$l \in [l_-, l_+]$ called "sub-gradient".

$f(x) - f(t) \ge l(x-t)$ $\forall x \in (a,b)$

Thm. (Jensen Inequality) Assume φ (not f!) is a convex function. and $a \le f(x) \le b$ $\forall x \in \Omega$

where $(\Omega, \mathcal{F}, \mu)$ is a measure space with $\mu(\Omega) = 1$.

Then. $\varphi(\mathbb{E} f) \le \mathbb{E}(\varphi(f))$ $\Leftrightarrow$ $\varphi\left(\int_\Omega f\, d\mu\right) \le \int_\Omega \varphi(f)\, d\mu$

RMK. 1) In Probability, $\mathbb{E} f = \int_\Omega f\, d\mu$. $(\mu(\Omega) = 1)$ "average of f with weight $d\mu$".

2) $f \in \mathcal{F}$, φ cts $\Rightarrow$ $\varphi(f) \in \mathcal{F}$. measurable.

Proof. (Background: φ convex $\Rightarrow$ $\partial f = [\ell_-, \ell_+] \neq \phi$, $\ell \in \partial f(x_0) \Rightarrow f(x) \geq f(x_0) + \ell(x - x_0)$)

φ convex $\Rightarrow$ $\varphi(z) \geq \varphi(\mathbb{E}f) + \ell(z - \mathbb{E}f)$ for $\forall\, z \in (a,b)$.

put $z = f(\omega) \Rightarrow$ $\varphi(f) \geq \varphi(\mathbb{E}f) + \ell(f - \mathbb{E}f)$. Taking $\mathbb{E}$ on both sides:

$$\mathbb{E}(\varphi(f)) = \int_\Omega \varphi(f)\,d\mu \geq \int_\Omega \varphi(\mathbb{E}(f)) + \ell(f - \mathbb{E}f)\,d\mu$$

$$= \mathbb{E}(\underbrace{\varphi(\mathbb{E}(f))}_{\text{const.}}) + \mathbb{E}(\ell(f - \mathbb{E}f))$$

$$\left.\begin{aligned} &= \varphi(\mathbb{E}(f)) + \ell(\mathbb{E}(f - \mathbb{E}f)) \\ &= \varphi(\mathbb{E}(f)) + \ell(\underbrace{\mathbb{E}(f) - \mathbb{E}(\underbrace{\mathbb{E}(f)}_{\text{const}})}_{=0}) \end{aligned}\right\} \text{linearity of } \mathbb{E}.$$

$$= \varphi(\mathbb{E}(f)). \qquad \blacksquare$$

Warning: make sure $\mu(\Omega) = 1$! In general, if $\mu(X) < +\infty$, it becomes

$$\varphi\left(\frac{\int_X f\,d\mu}{\mu(X)}\right) \leq \frac{\int_X \varphi(f)\,d\mu}{\mu(X)}$$

Example. $\varphi(x) = e^x$ convex. $\Rightarrow$ $\exp\left(\int_\Omega f\,d\mu\right) \geq \int_\Omega e^f\,d\mu$.

① In particular, if $\Omega = \{p_1, \cdots, p_n\}$ $\mu(\{p_i\}) = \frac{1}{n}$ $\forall i$. $f(p_i) = x_i$

it becomes $e^{\frac{\sum x_i}{n}} \leq \frac{1}{n}\left(\sum e^{x_i}\right)$

② define $y_i = e^{x_i}$, then it becomes

$$\left(\prod_{i=1}^n y_i\right)^{\frac{1}{n}} \leq \frac{\sum y_i}{n}. \quad \text{(AM-GM inequality)}$$

③ $\mu(\{p_i\}) = \alpha_i$, $0 \leq \alpha_i \leq 1$, $\sum \alpha_i = 1$. then

$$\prod_{i=1}^n y_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_i y_i$$

④ Young's inequality $a, b \geq 0$. $ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$ if $\frac{1}{p} + \frac{1}{q} = 1$ $p, q > 1$.

Prop. $(X, \mathcal{F}, \mu)$ measure space. $f, g \in \mathcal{F}$ with range $[0, +\infty]$

Let $1 < p, q < \infty$. $\frac{1}{p} + \frac{1}{q} = 1$ (p, q are conjugate)

1) Thm (Hölder) $\int_X fg\, d\mu \le \left(\int_X f^p d\mu\right)^{\frac{1}{p}} \left(\int_X g^q d\mu\right)^{\frac{1}{q}}$

2) Thm. (Triangle) $\int_X (f+g)^p d\mu \le \left(\int_X f^p d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p d\mu\right)^{\frac{1}{p}}$ Corollary: $\|\sum_{i=1}^N f_i\|_p \le \sum_{i=1}^N \|f_i\|_p$

(RMK: $\left(\int f^p d\mu\right)^{\frac{1}{p}} := \|f\|_p$.

Proof. 1) W.L.O.G. Assume $0 < \underbrace{\boxed{\int_X f^p d\mu}}_{:=A^p}, \underbrace{\int_X g^q d\mu}_{:=B^q} < +\infty$ (otherwise trivial)

$\Leftrightarrow \int_X fg\, d\mu \le AB \Leftrightarrow \int_X \frac{f}{A}\frac{g}{B} d\mu \le 1$ Because of this, we could Normalize f and g:

$\tilde{f} := \frac{f}{A}$ $\tilde{g} := \frac{g}{B}$. $\int_X \tilde{f}\tilde{g}\, d\mu \le 1$ with $\int \tilde{f}^p d\mu = \int \tilde{g}^q d\mu = 1$.

Using Young's, $\tilde{f}(x)\tilde{g}(x) \le \frac{1}{p}\tilde{f}^p(x) + \frac{1}{q}\tilde{g}(x)^q$, for each x.

$\Rightarrow \int \tilde{f}\tilde{g}\, d\mu \le \int \frac{1}{p}\tilde{f}^p d\mu + \int \frac{1}{q}\tilde{g}^q d\mu$

(By normalization) $\frac{1}{p} + \frac{1}{q} = 1$. ■

RMK $\int fg\, d\mu = \int \lambda f \cdot \lambda^{-1} g\, d\mu$

$\le \int \frac{1}{p}(\lambda f)^p d\mu + \int \frac{1}{q}(\lambda g)^q d\mu$.

2). See Youtube IMPA Real analysis.

$\int (f(x)+g(x))^p d\mu \le \left(\int (f+g)^p d\mu\right)^{\frac{p-1}{p}} \left[\left(\int f^p d\mu\right)^{\frac{1}{p}} + \left(\int g^p d\mu\right)^{\frac{1}{p}}\right]$

RMK. 1) if $0 < p < 1$, $\|f+g\|_p \le C_p(\|f\|_p + \|g\|_p)$. Typically $C_p > 1$.

2) The Hölder equality holds when "f^p is parallel to g^p":

$\exists\, \alpha, \beta$. $\alpha f^p = \beta g^p$ ($\int f^p d\mu < \infty$, $\int g^p d\mu < \infty$)

3) The Minkowski equality holds if

Def. L_p Space. Let $(X, \mathcal{A}, \mu)$ be a measure space. $0 \le p \le +\infty$. $f: X \to \mathbb{C}$.

$\|f\|_{L^p} = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}$.

$L^p(X) = \{ f \mid \|f\|_{L^p} < +\infty \}$

RMK. ℓ^p space. Suppose A is a set. μ is counting measure on A. $f: A \to \mathbb{C}$.

$$\|f\|_{\ell^p} = \left(\sum_i f(x_i)^p \right)^{\frac{1}{p}} \quad A = \{x_1, x_2, \dots\}$$

When $p = +\infty$: Essential Supremum. Suppose $g: X \to [0, +\infty]$ measurable. Let S be the set of $\alpha \in \mathbb{R}$ s.t. $\mu(|g|^{-1}(\alpha, +\infty)) = \mu(\{x \in X : |g(x)| > \alpha\}) = 0$.

If $S = \phi$, define $\|g\|_\infty := +\infty$.

If $S \ne \phi$, define $\|g\|_\infty = \inf S$. The inf is actually the minimum value of S: $S = [\inf S, +\infty)$

Then, $|g(x)| < \lambda$ iff $\lambda \ge |g|_\infty$.

Eg. $L^\infty(\mathbb{R}^k) = \{ f : \|f\|_\infty < +\infty \}$ essentially bounded w.r.t. Lebesgue measure

$\ell^\infty(A) = \{ f : \|f\|_{\ell^\infty} < +\infty \} \Leftrightarrow f(x_i)$ bounded.

Thm. (Hölder, Minkowski)

$1 \le p \le +\infty$. $q = \begin{cases} \frac{p}{p-1} & 1 < p < +\infty \\ \infty & p = 1 \\ 1 & p = +\infty \end{cases}$ conjugate of p. Then.

1) $\|fg\|_1 \le \|f\|_p \|g\|_q$

2) $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

RMK. Note. $\|\cdot\|_p$ is "not a norm" in the sense that $\|f\|_p = 0 \Rightarrow f = 0$ a.e. but not $f \equiv 0$.

So we introduce the equiv. class $f \sim g$ if $f = g$ a.e.

$L^p(X, \mu) = \{ [f] \mid \|f\|_p < +\infty \}$. Now $\|\cdot\|_p$ is indeed a norm.

equiv. class of f.

Thm $L^p(X,\mu)$ is a Banach Space for $1 \le p \le +\infty$ (Completeness)

Proof. 1) $p=+\infty$ Suppose $\{f_n\}_{n=1}^{\infty}$ is Cauchy in L^∞. $A_k := \{x : |f_k(x)| > \|f_k\|_\infty\}$

$B_{mn} := \{x : |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\}$

So $\mu(A_k) = \mu(B_{m,n}) = 0$.

$E := \left(\bigcup_{k\ge 1} A_k\right) \cup \left(\bigcup_{m,n\ge 1} B_{mn}\right)$ has zero measure. (Bad sets)

On E^c, $f_n(x)$ conv. unif (in $\mathbb{R}$ or $\mathbb{C}$) to a bounded function $f(x)$.

Define $f(x)=0$ $\forall x \in E$. Then $\|f_n\|_\infty \to \|f\|_\infty$, $\|f\|_\infty < +\infty \Rightarrow f \in L^\infty$.

2) $1 \le p < +\infty$:

Lemma. (Completeness of normed spaces) A normed vector space X is complete iff every absolutely conv. series in X converges.

proof. $\Rightarrow$: X completed and $\sum_{n=1}^{\infty} \|x_n\| < +\infty$. $S_N = \sum_{n=1}^{N} \|x_n\|$. $\uparrow$. bounded. So converges (in $\mathbb{R}$) Call $s := \sum_{i=1}^{\infty} \|x_n\|$

$\|S_N - S\| = \|\sum_{n=N+1}^{\infty} x_n\| \le \sum_{n=N+1}^{\infty} \|x_n\| < \varepsilon$ for large N. So $S_N \to S$ by completeness of X.

$\Leftarrow$: Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy. Choose a subsequence s.t. $n_1 < n_2 < \dots$

$\|x_m - x_n\| < 2^{-j}$ for $\forall\, m,n \ge n_j$. (by Cauchy).

Then $\{x_{n_j}\}$ converges. say $x_{n_j} \to L$ in X.

Now show the whole sequence converges by Cauchy.

Return to $1 \le p < +\infty$. it's sufficient to show $\sum_{n=1}^{\infty} \|g_n(x)\|_p < +\infty$ then $\sum_{n=1}^{\infty} g_n(x)$ conv. in L^p.

Consider $G_N(x) = \sum_{n=1}^{N} |g_n(x)|$ $G := \sum_{n\ge 1} |g_n(x)|$.

$\|G_N(x)\|_p \overset{\text{triangle}}{\le} \sum_{n=1}^{N} \|g_n\|_p \le \sum_{n=1}^{\infty} \|g_n(x)\| < +\infty$

$\int |G|^p d\mu = \int \lim_{N\to\infty} |G_N|^p d\mu \overset{\text{MCT}}{=\!=} \lim_{N\to\infty} \int |G_N|^p d\mu < +\infty$

Then $\sum_{n=1}^{\infty} g_n(x)$ conv. a.e. Set $\tilde{g} = \begin{cases} \sum_{n=1}^{\infty} g_n(x) & \text{if conv.} \\ 0. & \text{if diverges.} \end{cases}$ then $|\tilde{g}(x)| \le G(x)$, $\tilde{g} \in L^p$.

then $|\tilde{g} - \sum_{n=1}^{N} g_n| \le 2|G(x)|$. Hence $\sum_{n=1}^{N} g_n \to \tilde{g}$ in L^p by DCT.

Prop (Embedding) $0<p<q\leq+\infty$, $\ell^p(X)\subseteq\ell^q(X)$ and $\|f\|_{\ell^p}\geq\|f\|_{\ell^q}$

Ref. Folland 实分析 little ℓ^p.

proof. 1) $q=+\infty$. $\|f\|_\infty^p=\sup_{\alpha\in X}|f(\alpha)|^p\leq\sum_{\alpha\in X}|f(\alpha)|^p$.

$\Rightarrow \|f\|_\infty\leq\|f\|_{\ell^p}$.

2) for $q<+\infty$: Use interpolation 插值.

Prop. Suppose $0<p<q<r\leq+\infty$. Then $L^p(X)\cap L^r(X)\subseteq L^q(X)$

$\|f\|_q\leq\|f\|_p^\lambda\|f\|_r^{1-\lambda}$. where λ satisfying $\frac{1}{q}=\frac{\lambda}{p}+\frac{1-\lambda}{r}$.

RMK. 对数凸性 $\log\|f\|_q\leq\lambda\log\|f\|_p+(1-\lambda)\log\|f\|_r$.

proof. $\int|f|^q d\mu=\int|f|^{\lambda q}|f|^{(1-\lambda)q}d\mu\underset{\text{Hölder}}{\leq}\||f|^{\lambda q}\|_{\frac{p}{\lambda q}}\||f|^{(1-\lambda)q}\|_{\frac{r}{(1-\lambda)q}}$ $\quad\left(\frac{\lambda q}{p}+\frac{1-\lambda}{r}q=1\right)$

$=\|f\|_p^{\lambda q}\|f\|_r^{(1-\lambda)q}$. (Hint: $\||f|^a\|_p=\|f\|_{ap}^a$ $\forall a>0$.)

Prop. Suppose $\mu(X)<+\infty$. $0<p<q\leq+\infty$ then

$L^p(X)\supseteq L^q(X)$ $\|f\|_p\leq\|f\|_q\cdot\mu(X)^{\frac{1}{p}-\frac{1}{q}}$

proof. 1) $q=+\infty$. trivial

$\|f\|_p^p=\int_X|f|^p d\mu\leq\|f\|_\infty^p\mu(X)$.

2) $q<+\infty$. $\|f\|_p^p=\int\underbrace{|f|^p}_{L^{\frac{q}{p}}}\cdot\underbrace{1}_{L^{\frac{q}{q-p}}}d\mu\underset{\text{Hölder}}{\leq}\||f|^p\|_{\frac{q}{p}}\|1\|_{\frac{q}{q-p}}$

$\leq\|f\|^p\cdot\mu(X)^{\frac{q-p}{q}}$ ∎

Prop. density of simple functions

① $1 \le p < +\infty$. the set of simple functions $f = \sum_{i=1}^{\infty} \alpha_i \mathbb{1}_{E_i}$ where $\mu(E_j) < +\infty$ is dense in L^p.

② if $p = +\infty$. simple functions are dense in L^∞.

Proof. ① pick $f \in L^p$. by using approximation by simple functions. we could find a sequence $\{f_n\}_{n=1}^{\infty}$ simple functions

$f_n \to f$ a.e. $|f_n| \le |f|$. $|f_n| \uparrow$ to $|f|$.

So $f_n \in L^p$. $|f_n - f|^p \le \underbrace{2^p |f|^p}_{控制函数} \in L^1$

$\|f_n - f\|_p \to 0$ as $n \to \infty$. (by DCT).

$f_n = \sum_{j=1}^{N} a_j \mathbb{1}_{E_j}$ $a_j \neq 0$. E_j disjoint.

$\int |f_n|^p d\mu = \sum_{i=1}^{N} |a_j|^p \mu(E_j) < +\infty \Rightarrow \mu(E_j) < +\infty$

② By using approximation $f_n \to f$ a.e. $|f_n| \le |f|$. $|f_n| \to |f|$.

So $\|f_n\|_\infty \le \|f\|_\infty$, $\|f_n\| \in L^p$.

Thm. Chebyshev inequality

$f \in L^p$. $0 < p < +\infty$ $\forall \alpha > 0$. $\mu(x : |f(x)| > \alpha) \le \left(\frac{\|f\|_p}{\alpha}\right)^p$.

Proof. $\|f\|_p^p = \int_X |f|^p d\mu = \left(\int_{|f|>\alpha} + \int_{|f| \le \alpha}\right) |f|^p d\mu$

$$\ge \int_{|f|>\alpha} |f|^p d\mu \ge \alpha^p \mu(|f| > \alpha).$$

RMK Weak L^p spaces

$$\|f\|_{L^p\text{-weak}}^p = \sup_{\alpha > 0} \alpha^p \mu(|f| > \alpha)$$

Prop. The following classes of functions are dense in L^1. (Stein)

(i) Simple function (ii) Step functions (iii) smooth compact support functions

$$\overline{C_c^\infty(X)} = L^1(X)$$