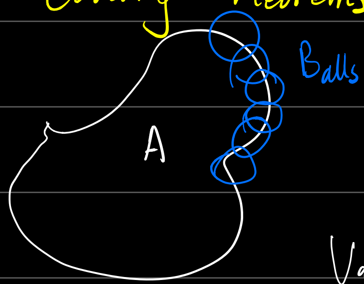
LEBESGUE DIFF. THM.

1 Covering Theorems.



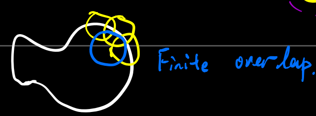
$A \subseteq \bigcup_{j \in J} B_j$

Two contradictory task:

- make as "disjoint" as possible
- same radius.

Vatali covering: 牺牲第二个. enlarge the balls.

Bescovitch covering. 牺牲第一个, 有重叠



Thm. Vatali covering, finite version. 3 ball or 5 ball.

Suppose $W \subseteq \bigcup_{i=1}^{N} B(x_i, r_i)$ Then $\exists\, S \subseteq \{1, 2, \dots, N\}$, s.t.

- $\bigcup_{i \in S} B(x_i, r_i)$ are disjoint.
- $W \subseteq \bigcup_{i \in S} B(x_i, 3r_i)$ some book choose 5. 大于2即可

Proof. Greedy-Algorithm. Always Pick the largest ball (Note: finite balls in total)

Step 1: Pick $r^1 = \max\{r_i : 1 \le i \le N\}$. say $r^1 = r_{n_1}$

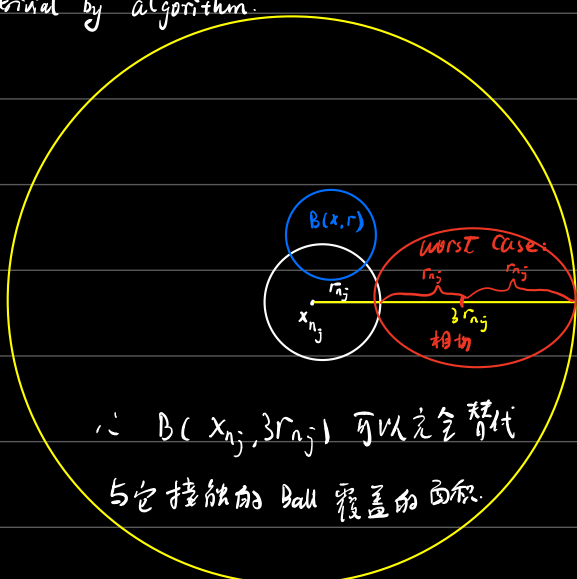
2: Delete all balls which intersect with $B(x_{n_1}, r_{n_1})$

3. Pick second largest ball in the remaining and repeat.

By finiteness, the algorithm will end with $B(x_{n_j}, r_{n_j})$ $1 \le j \le s. \le N$. $S = \{n_1, n_2, \dots n_s\} \subseteq \{1, 2, \dots N\}$

Claim: $\bigcup_{j \in S} B(x_{n_j}, r_{n_j})$ is the desired covering.

- Disjoint: trivial by algorithm.
- covering:



$\therefore B(x_{n_j}, 3r_{n_j})$ 可以完全替代与它接触的 Ball 覆盖的面积.

Def. 1) A collection $\mathcal{F}$ of closed balls in $\mathbb{R}^n$ is a *cover* of a set $A \subseteq \mathbb{R}^n$ if $A \subseteq \bigcup_{B \in \mathcal{F}} B$

2) $\mathcal{F}$ is called a "*fine cover*" of A if in addition, $\inf\{\operatorname{diam} B \mid x \in B,\ B \in \mathcal{F}\} = 0$, $\forall x \in A$.

OR "Vatali Cover"

Theorem. (Vatali. infinite version)

Let $\mathcal{F}$ be any collection of nondegenerate closed balls in $\mathbb{R}^n$. $\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < +\infty$. Then,

$\exists$ a countable subcollection $\mathcal{G} \subseteq \mathcal{F}$ s.t.

- Disjoint: $B_i \cap B_j = \phi$. $B_i, B_j \in \mathcal{G}$.
- $\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} (5B)$

取3即可

Proof. 模仿 finite case. 贪心算法.

Step 1. Denote $D := \sup_{B \in \mathcal{F}} \operatorname{diam} B < +\infty$. Set $\mathcal{F}_j = \left\{ B \in \mathcal{F} \,\middle|\, \frac{D}{2^j} < \operatorname{diam} B \leq \frac{D}{2^{j-1}} \right\}$, $j = 1, 2, \cdots$.

若用 $\frac{D}{c^j}$ 分割，
则最终结果放大系数为 $1+2c$.

Now define $\mathcal{G}_j \subseteq \mathcal{F}_j$ as follows:

$j=1$: $\mathcal{F}_1 = \{B \in \mathcal{F} \mid \frac{D}{2} < \operatorname{diam} B \leq D\}$. Let $\mathcal{G}_1$ be any maximal disjoint countable sub-collection of $\mathcal{F}_1$? why countable?

Assume $\mathcal{G}_1, \cdots, \mathcal{G}_{n-1}$ has been chosen. We then chose $\mathcal{G}_n$ be any maximal disjoint sub-collection of

Don't want any intersection with previous guys.

$$\left\{ B \in \mathcal{F}_n \,\middle|\, B \cap B' = \phi \text{ for all } B' \in \bigcup_{k=1}^{n-1} \mathcal{G}_k \right\}$$

? why countable?

Take $\mathcal{G} := \bigcup_{j \geq 1} \mathcal{G}_j$ countable since each $\mathcal{G}_j$ is countable (?)

Solution: each ball in $\mathcal{G}$ contains an rational point $\mathbb{Q}^d$.
By disjointness, different balls have diff. rationals.
$\Rightarrow$ At most countable.

- disjoint: trivial.
- covering property: Claim: For each $B \in \mathcal{F}$, $\exists$ $B' \in \mathcal{G}$, $B \cap B' \neq \phi$. $B \subseteq 5B'$.

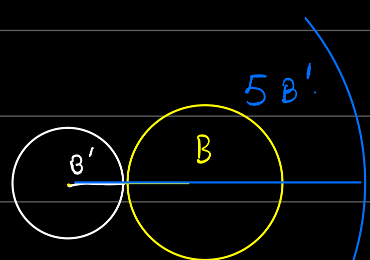
proof of the claim: Fix $B \in \mathcal{F}$. $\exists j$, $B \in \mathcal{F}_j$. By the maximality of $\mathcal{G}_j$,

$\exists$ a ball $B' \in \mathcal{G}_j$. $B' \in \bigcup_{k=1}^{j} \mathcal{G}_k$ s.t. $B' \cap B \neq \phi$.

Note that $\operatorname{diam} B' \geq \frac{D}{2^j}$, $\operatorname{diam} B \leq \frac{D}{2^{j-1}}$

$\Rightarrow \operatorname{diam} B \leq 2 \operatorname{diam} B'$.

$\Rightarrow B \subseteq 5B'$.



Thm. variant of Vatali.

Assume $\mathcal{F}$, collection of balls, is a fine cover of A by closed balls. $\sup\limits_{B\in\mathcal{F}} \operatorname{diam} B < +\infty$ Then there exists a countable subset G of $\mathcal{F}$ s.t. for each finite subset $\{B_1, \dots, B_n\} \subseteq \mathcal{F}$.

$$A \setminus \bigcup_{k=1}^{n} B_k \subseteq \bigcup_{B\in G\setminus\{B_1,\dots,B_n\}} 5B$$

Proof. We first choose a sub-collection G the same as proof of Vatali Thm.

1) Case 1: $A \subseteq B_1 \cup B_2 \cup \dots \cup B_n$. we are done.

2) Case 2: Let $x \in A \setminus \bigcup_{k=1}^{n} B_n$. Since Balls in $\mathcal{F}$ are closed, $\mathcal{F}$ is a fine cover, there exists $B \in \mathcal{F}$ with $x \in B$ and $B \cap B_k = \phi$, $k = 1, 2, \dots, n$. But follows from our claim in previous proof, $\exists B' \in G$ $B \cap B' \neq \phi$. $B \subseteq 5B'$.

* Thm. (Filling open sets with balls)

Let $U \subseteq \mathbb{R}^n$ open. $\delta > 0$. $\exists$ countable collection G of disjoint closed balls in U s.t.

- $\operatorname{diam} B < \delta \quad \forall B \in G$.
- $\mu\left(U \setminus \bigcup_{B\in G} B\right) = 0$.

n-dim Lebesgue

Proof. 1. Fix $1 - \frac{1}{5^n} < \theta < 1$. Assume $\mu(U) < +\infty$

2. Claim. $\exists$ a finite covering $\{B_i\}_{i=1}^{M_1}$ of disjoint closed balls in U. s.t. $\operatorname{diam} B_i < \delta$, $\mu\left(U \setminus \bigcup_{i=1}^{M} B_i\right) \leq \theta\, \mu(U)$.

Justification of the claim: Let $\mathcal{F}_1 = \{ B \subseteq U \mid \operatorname{diam} B < \delta \}$ By Vatali. $U \subseteq \bigcup_{B \in G_1} 5B$. $G_1 \subseteq \mathcal{F}$

$\Rightarrow \mu(U) \le \mu \sum_{B \in G_1} \mu(5B) = 5^n \sum_{B \in G_1} \mu(B) \overset{\text{disjoint of } G_1}{=\!=} 5^n \mu\left(\bigcup_{B \in G_1} B\right)$

$\Rightarrow \mu\left(\bigcup_{B \in G_1} B\right) \ge 5^{-n} \mu(U)$.

$\mu(U \setminus \bigcup_{B \in G_1} B) \le (1 - 5^{-n}) \mu(U) < \theta \mu(U)$.

G_1 countable $\Rightarrow \mu(U \setminus \bigcup_{j=1}^{M_1} B_j) \le \theta \mu(U)$

3. Iterate. $U_2 = U_1 - \bigcup_{i=1}^{M_1} B_i$ (still open) key reason why pass countable union of closed sets to finite union.

$\mathcal{F}_2 = \{ B \mid \operatorname{diam} B < \delta.\ B \subseteq U_2 \}$ $\Rightarrow \exists$ finite disjoint closed balls $\{ B_{M_1+1}, B_{M_1+2}, \cdots, B_{M_2} \}$ s.t.

$\mu(U - \bigcup_{i=1}^{M_2} B_i) \overset{?}{=} \mu(U_2 \setminus \bigcup_{i=M_1}^{M_2} B_i) \le \theta\ (U_2) \le \theta^n \mu(U)$

重复上述过程得 $\mu(U \setminus \bigcup_{i=1}^{M_k} B_i) \le \theta^k \mu(U)$. $\Rightarrow \mu(U - \bigcup_{i=1}^{\infty} B_i) = 0$.

4. If $\mu(U) = +\infty$: apply the above argument to $U^n = U \cap \{ n \le |x| < n+1 \}$

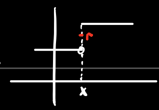
*Thm. Vatali covering (IMPA video version)

$A \subseteq \mathbb{R}^n$. (可能不可测). $\mathcal{F}$ is Vatali covering of A. then

- $\exists$ countable sub-collection $J \subseteq \mathcal{F}$.
- $\mu^*(A \setminus \bigcup_{B \in J} B) = 0$.
- $\forall \varepsilon > 0$. $\exists$ finite sub-collection $\{ B_1, \cdots, B_N \} \subseteq J$, $\mu^*(A \setminus \bigcup_{i=1}^{N} B_i) < \varepsilon$

#2. BV functions on $[a,b]$

We begin with a simple **theorem:** Suppose $f:[a,b]\to\mathbb{R}$ monotone increasing. Then f is cts a.e.

Proof: For each $x\in[a,b]$ s.t. f not cts, choose $r\in\mathbb{Q}$ s.t.  then 1-1 correspondance between $\mathbb{Q}$ and discontinuities. So cts a.e.

Definitions. Dini derivative. $D^+ := \overline{\lim_{h\to0^+}} \frac{f(x+h)-f(x)}{h}$, $D^- := \underline{\lim_{h\to0^+}} \frac{f(x+h)-f(x)}{h}$. $D_+ := \overline{\lim_{h\to0^+}}$, $D_- := \underline{\lim_{h\to0^-}} \ldots$

So $D^+ \geq D^-$, $D_+ \geq D^-$. f is diff. $\iff D^+ = D^- = D_+ = D^-$

Thm. $f:[a,b]\to\mathbb{R}$ monotone increasing. $E := \{x\in(a,b) \mid f \text{ is diff. at } x\}$. Then.

- E is measurable
- $\mu([a,b]\setminus E) = 0$. i.e. monotone increasing functions are a.e. differentiable.

Proof:

Def. Total Variation $V_a^b(f) := \sup_{P} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|$ where the sup is taking over all partitions of $[a,b]$.

If $V_a^b(f) < +\infty$, we say $f \in BV([a,b])$

Thm. $BV \Rightarrow$ Riemann integrable.

proof: $\sup_P \sum |f(x_{i+1}) - f(x_i)| < +\infty \Rightarrow$ Fix a partition P. $\omega(P,f) = \sum \left(\sup_{[x_{i+1},x_i]} f - \inf_{[\ ,\ ]} f \right) \Delta x_i \leq V_a^b(f) \cdot \|P\| < \varepsilon$ if we choose $\|P\| < \frac{\varepsilon}{V_a^b(f)}$.

Eg. $f: [a,b] \to \mathbb{R}$. If f is monotone increasing / decreasing and bounded, then $f \in BV[a,b]$. since

$$\sum |f(x_{i+1}) - f(x_i)| = \sum f(x_{i+1}) - f(x_i) = f(b) - f(a) < +\infty.$$

Eg. $f: [a,b] \to \mathbb{R}$ diff. at every point with bounded derivative: $|f'(x)| \leq M$ Then

$$\sum |f(x_{i+1}) - f(x_i)| = \sum |f'(c_i)| \Delta x_i \leq M(b-a). \quad \text{hence } f \in BV[a,b]$$

Eg. $F(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^a \sin(x^{-b}) & \text{if } x > 0. \end{cases}$ is of BV on $[0,1]$ iff. $a > b$.

* Basic observation: the more refinement of partition, the greater the variation. (Triangle ineq.)

Def. $T_F(x) := \sup_P \sum_{i=1}^{N} |F(t_i) - F(t_{i-1})|$ where $a = t_0 \leq t_1 \leq \cdots \leq t_N = x$. is the total Variation of f on $[a,x]$. Similarly. $P_F(x) = \sup_P \sum_{+} F(t_i) - F(t_{i-1})$. where we take sum of all terms that $F(t_i) > F(t_{i-1})$.

$N_F(x) = \sup_P \sum_{-} -(F(t_i) - F(t_{i-1}))$ where we take sum of all terms that $F(t_i) < F(t_{i-1})$.

类比 $f = f^+ - f^-$

Note: $N_F(x) > 0$.

Lemma. If $F \in BV[a,b]$. Then $P_F(x) + N_F(x) = T_F(x)$

$F(x) - F(a) = P_F(x) - N_F(x)$ for all $x \in [a,b]$

Proof. $\forall \varepsilon > 0$. $\exists P, P'$ partitions of $[a,b]$ s.t. $|P_F(x) - \underbrace{\sum_{P:+} F(t_i) - F(t_{i-1})}_{\text{depend on } P}| < \varepsilon$ and $|N_F(x) - \sum_{P':-} \underbrace{-(F(t_i) - F(t_{i-1}))}_{\text{related to } P'}| < \varepsilon$.

Take $Q = P \cup P'$. common refinement, Then

$|P_F(x) - \sum_{Q:+} F(t_i) - F(t_{i-1})| < \varepsilon$ and $|N_F(x) - \sum_{Q:-} -(F(t_i) - F(t_{i-1}))| < \varepsilon$. both hold.

$$f(x) - f(a) = \sum_{Q} F(t_i) - F(t_{i-1}) = \sum_{Q:+} F(t_i) - F(t_{i-1}) - \sum_{Q:-} -(F(t_i) - F(t_{i-1}))$$

$$\Rightarrow \left| f(x) - f(a) - (P_F(x) - N_F(x)) \right| \leq \left| \sum_{Q:+} F(t_i) - F(t_{i-1}) - P_F(x) \right| + \left| N_F(x) - \sum_{Q:-} -(F(t_i) - F(t_{i-1})) \right| < 2\varepsilon$$

So $f(x) - f(a) = P_F(x) - N_F(x)$.

For all x and partition P, $\sum |F(t_i) - F(t_{i-1})| = \sum_{+} + \sum_{-}$, taking sup on both sides. done.

Thm. $F: [a,b] \to \mathbb{R}$ is of bounded variation if and only if F is the difference of two (bounded) increasing functions.

Proof. $\Leftarrow$: trivial

$\Rightarrow$ if $F \in BV$, then $F_1(x) = f(a) + P_F(x)$, $F_2(x) = f(a) + N_F(x)$ are desired decomposition. ∎

* Corollary: As a immediate consequence, we get BV functions are diff. a.e. Ref: Royden. 4th e.

#3 Lebesgue differentiation Thm.

Def. Locally integrable. say f is locally integrable if. $\forall$ ball $B \subseteq \mathbb{R}^n$, $\int_B f(x)dx < +\infty$ denote by $f \in L^1_{loc.}(\mathbb{R}^n)$

OR Equivalently. $\int_K f\,dx < +\infty$ $\forall$ compact $K \subseteq \mathbb{R}^n$

Thm.* Suppose $f \in L^1_{loc}(\mathbb{R}^n)$ then for a.e. $x \in \mathbb{R}^n$.

$$\lim_{r\to 0^+} \frac{\int_{B(x,r)} |f(x)-f(y)|\,dy}{\mu(B(x,r))} = 0.$$

where μ is the Lebesgue measure on $\mathbb{R}^n$.

Eg. 1 dimension case. $\lim\limits_{r\to 0^+} \frac{1}{2r}\int_{x-r}^{x+r} |f(x)-f(y)|\,dy = 0.$ a.e. $x \in \mathbb{R}$.

$\Rightarrow \lim\limits_{r\to 0^+} \frac{1}{2r}\int_{x-r}^{x+r} f(y)dy = f(x)$.

$\Leftrightarrow$ f integrable, $F(x) := \int_a^x f(y)dy$ then F differentiable and $F'(x)=f(x)$ a.e. 微积分第一基本定理.

RMK. points where such property holds are called "Lebesgue point".

key: use maximal function to control the local oscillation.

Def. Maximal function $f^*(x) = \sup\limits_{x\in B} \frac{1}{\mu(B)}\int_B |f(y)|\,dy$, here x may not be the center of the balls.

Centered version: $f^*_c(x) = \sup\limits_{r>0} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)|\,dy = \sup\limits_{r>0} \left(\mathbb{1}_{B(x,r)} \overset{卷积}{*} |f|\right)(x)$.

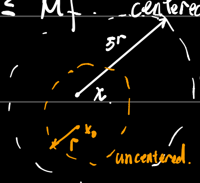
(c: centered)

RMK. centered and uncentered maximal function are comparable (equivalent): $C_1 (\overset{c.}{M}f)(x) \le \overset{uc.}{M}f(x) \le C_2 \overset{c}{M}f(x)$.

where C_1, C_2 are constant depend on the dimension $\mathbb{R}^n$. We could choose either one freely.

Proof: Obviously $\overset{(c)}{M}f \le \overset{(uc)}{M}f$. centered is a special case of uncentered. For the other side. cover the uncentered ball

by $B(x, 5r)$:



Then some constant depending on the dimension comes up.

Lemma Suppose f is integrable on $\mathbb{R}^n$. then

(1) Mf is measurable (2) $Mf(x) < +\infty$ a.e.

proof: Call $E_\alpha := \{x \in \mathbb{R}^n \mid f^*(x) > \alpha\}$. (1) $x \in E_\alpha \Leftrightarrow \sup\limits_{x\in B} \frac{1}{\mu(B)}\int_B |f(y)|dy > \alpha \Leftrightarrow \exists B$. containing x, $\frac{1}{\mu(B)}\int_B |f(y)|dy > \alpha$ $(*)$

Then for x' close enough to x, $x' \in B$. $(*)$ also holds $\Rightarrow x' \in E_\alpha \Rightarrow E_\alpha$ is open. $\Rightarrow Mf$ is measurable.

(2) 由下面直接得. $\{x \mid f^*(x) = +\infty\} \subseteq E_\alpha$. $\forall \alpha > 0$. $\longrightarrow 0$ as $\alpha \to +\infty$.

Lemma. the maximal function M maps L^1 to weak L^1, and L^p to L^p if $1<p\le+\infty$. i.e.

$\mu(\underbrace{x\in\mathbb{R}^n: f(x)>\alpha}_{:=E_\alpha}) < \frac{C}{\alpha}\|f\|_{L^1}$ $\|Mf\|_{L^p}\le C_p\cdot\|f\|_{L^p}$ for some constant C, C_p. depending on dimension.

Proof: (1) L^1 to weak L^1.

Consider a compact subset K of E_α. $K\subseteq\bigcup_{x\in E_\alpha} B_x$ where B_x satisfies $\frac{1}{\mu(B_x)}\int_{B_x}|f(y)|dy>\alpha \Longleftrightarrow \mu(B_x)<\frac{1}{\alpha}\int_{B_x}|f(y)|dy$

$\exists$ finite subcollection $\{B_{x_1},\cdots,B_{x_N}\}$ covers K. by compactness.

$$\mu(K)\le\mu\left(\bigcup_{i=1}^N B_{x_i}\right)\le\sum_{i=1}^N\mu(B_{x_i})\le\sum_{i=1}^N\frac{1}{\alpha}\int_{B_x}|f(y)|dy=\sum_{i=1}^N\frac{1}{\alpha}\int_{\mathbb{R}^n}|f(y)|dy=\frac{N}{\alpha}\|f\|_{L^1}.$$

Warning: It does not work since this "N" depends on α

By Vatali covering (finite version): $\exists\{B_{x_{j_1}},\cdots B_{x_{j_M}}\}\subseteq\{B_{x_1},\dots,B_{x_N}\}$, disjoint. & $K\le\bigcup_{i=1}^N B_{x_i}\le\bigcup_{k=1}^M 3B_{x_{j_k}}$

$$\Rightarrow \mu(K)\le\mu\left(\bigcup_{k=1}^M 3B_{x_{j_k}}\right)\overset{B_{x_{j_k}}\text{ disjoint}}{\le}\sum_{k=1}^M 3^n\mu(B_{x_{j_k}})\le\frac{3^n}{\alpha}\sum_{k=1}^M\int_{B_{x_{j_k}}}|f(y)|dy\underset{B_{j_k}\text{'s disjoint}}{\le}\frac{3^n}{\alpha}\int_{\bigcup B_{x_{j_k}}}|f(y)|dy\le\frac{3^n}{\alpha}\|f\|_{L^1}.$$

Since $\mu(K)\le\frac{C}{\alpha}\|f\|_{L^1}$ for all compact subset K of E_α. It also holds for E_α. ∎

(2) L^p to L^p. Idea: almost doing Marcinkiewicz interpolation Thm. 见附录.

Fix $p\in(1,+\infty)$. $p=+\infty$ is easy.

$\alpha>0$. $f_1(x)=\begin{cases}f(x) & \text{if } |f|>\frac{\alpha}{2}\\ 0 & \text{otherwise}.\end{cases}$ Obviously $Mf(x)\le Mf_1(x)+\frac{\alpha}{2}$ $\Rightarrow \mu(Mf>\alpha)\le\mu(Mf>\frac{\alpha}{2})$.

"Layer-Cake" representation of L^p-norm of f: $\int_{\mathbb{R}^n}|f|^p dx=\int_{\mathbb{R}^n}\left(\int_0^\infty \mathbb{1}_{|f|^p>t^p}dt\right)dx.$

$$=\int_{\mathbb{R}^n}\left(\int_0^\infty \mathbb{1}_{|f|>t}\, p t^{p-1}dt\right)dx$$

$$\therefore\ \|f\|_{L^p}^p \overset{\text{Fubini}}{=} \int_0^\infty \mu(|f|>t)\, p t^{p-1}dt$$

$$\|Mf\|_{L^p}^p = p\int_0^\infty\mu(|Mf(x)|>t)\,t^{p-1}dt\le p\int_0^\infty\mu\left(|Mf(x)|>\frac{t}{2}\right)t^{p-1}dt$$

$$=\text{const.}\int_0^\infty\mu(|Mf_1|>t)\,t^{p-1}dt$$

$$\le\text{const.}\int_0^\infty\frac{\|f_1\|_{L^1}}{t}\,t^{p-1}dt.$$

$$=\text{const.}\int_0^\infty\frac{1}{t}\left(\int_{|f(x)|>\frac{t}{2}}|f(x)|dx\right)t^{p-1}dt$$

$$\overset{\text{Fubini again}}{\le}\text{const.}\int_{\mathbb{R}^n}|f(x)|^p dx=\text{const.}\|f\|_{L^p}^p.$$ ∎

We prove first a stronger version of Lebesgue differentiation Thm. (Stein)

(Global) Suppose f is (globally) integrable i.e. $f \in L^1(\mathbb{R}^n)$. Then $\lim_{\substack{\mu(B)\to 0\\ x\in B}} \frac{1}{\mu(B)} \int_B f(y)\,dy = f(x)$. a.e.

Proof. $E_\alpha = \left\{ x \in \mathbb{R}^n \;\middle|\; \limsup_{\substack{\mu(B)\to 0\\ x\in B}} \left| f(x) - \frac{1}{\mu(B)} \int_B f(y)\,dy \right| > 2\alpha \right\}$.

Then $E := \left\{ x \;\middle|\; \limsup \left| f(x) - \frac{1}{\mu(B)} \int_B f(y)\,dy \right| > 0 \right\} = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}}$. WANT: $\mu(E_\alpha) = 0$.

Approximate f by a compactly support smooth function g s.t. $\|f-g\|_{L^1} < \varepsilon$.

$$f(x) - \frac{1}{\mu(B)} \int_B f(y)\,dy = \underbrace{\left[f(x) - g(x)\right]}_{①} + \underbrace{\left[g(x) - \frac{1}{\mu(B)} \int_B g(y)\,dy\right]}_{②} + \underbrace{\left[\frac{1}{\mu(B)} \int_B g(y) - f(y)\,dy\right]}_{③} \quad (*)$$

$|LHS| \le |①| + |②| + |③|$.

Since g is cts at x, $\forall \varepsilon > 0\ \exists \delta > 0$ $|g(x) - g(y)| < \varepsilon$ if $|x-y| < \delta$

$\Rightarrow \left| \frac{1}{\mu(B)} \int_B g(y)\,dy - g(x) \right| = \left| \frac{1}{\mu(B)} \int_B g(x) - g(y)\,dy \right| \le \frac{1}{\mu(B)} \int_B |g(x) - g(y)|\,dy < \varepsilon \Leftrightarrow ② \to 0$ in $(*)$.

So take $\mu(B) \to 0$ in $(*)$: $|LHS| \le |f(x) - g(x)| + |M(g-f)(x)|$

Call $F_\alpha := \{ x \mid |f-g| > \alpha \}$. $G_\alpha := \{ M(g-f) > \alpha \}$. Then $E_\alpha \subseteq F_\alpha \cup G_\alpha$.

By Chebyshev: $\mu(F_\alpha) \le \frac{1}{\alpha} \int |f-g|\,d\mu = \frac{1}{\alpha} \|f-g\|_{L^1}$

By previous Lemma, $\mu(G_\alpha) \le \frac{A}{\alpha} \int |g-f|\,d\mu = \frac{A}{\alpha} \|f-g\|_{L^1}$

$\Rightarrow \mu(E_\alpha) \le \frac{1}{\alpha}(A+1) \|f-g\|_{L^1}$

By approximation Thm, $\|f-g\|_{L^1} < \varepsilon$. $\forall \varepsilon > 0$. So. $\mu(E_\alpha) = 0$.

$\Rightarrow \mu(E) = 0$. i.e. $\lim_{\substack{\mu(B)\to 0\\ x\in B}} \int_B f(y)\,dy = f(x)$ a.e. ■

We now proceed to locally integrable f. i.e. Thm *.

Proof of Thm *: Apply Thm (Global) on Closed ball containing x. Done. ■

Marcinkiewicz interpolation Thm.

T Sublinear Operator if : $|T(f+g)| \le C(|Tf|+|Tg|)$ eg. Maximal function operator M.

Suppose T is a sublinear operator which maps measurable functions to measurable functions s.t.

T: $L^{p_0} \longrightarrow$ weak L^{p_0}. $L^{p_1} \longrightarrow$ weak L^{p_1}. $0 < p_0 < p_1 \le +\infty$

Then T maps L^p to L^p $\forall\, p \in (p_0, p_1)$ with diverging constants as $p \to p_0$ or $p \to p_1$

Proof: For simplicity, assume $|T(f+g)| \le |Tf| + |Tg|$. $f, g \ge 0$

Let $\alpha > 0$, $t > 0$. $f(x) = f^\alpha(x) + f_\alpha(x)$ where $f^\alpha(x) = \begin{cases} f(x) & \text{if } f(x) \ge \frac{\alpha}{2} \\ 0 & \text{otherwise} \end{cases}$ $f_\alpha(x) = \begin{cases} 0 & \text{otherwise} \\ f(x) & \text{if } 0 \le f(x) < \frac{\alpha}{2} \end{cases}$

$$\|Tf\|_{L^p}^p = p\int_0^\infty \mu(Tf > \alpha)\, \alpha^{p-1}\, d\alpha$$

$$\{Tf > \alpha\} = \{T(f^\alpha + f_\alpha) > \alpha\} \subseteq \{Tf^\alpha > \tfrac{\alpha}{2}\} \cup \{Tf_\alpha > \tfrac{\alpha}{2}\}$$

$$\Rightarrow \mu(Tf > \alpha) \le \mu(Tf^\alpha > \tfrac{\alpha}{2}) + \mu(Tf_\alpha > \tfrac{\alpha}{2})$$

Notation $\lesssim$: $X \lesssim Y \Leftrightarrow X \le \text{Const. } Y$

$$\|Tf\|_{L^p}^p \lesssim \int_0^\infty \mu(Tf^\alpha > \tfrac{\alpha}{2})\alpha^{p-1}\, d\alpha + \int_0^\infty \mu(Tf_\alpha > \tfrac{\alpha}{2})\, \alpha^{p-1}\, d\alpha$$

第一项: by weak $-L^{p_0}$ boundedness. $\lesssim \int_0^\infty \frac{1}{(\frac{\alpha}{2})^{p_0}} \|f^\alpha\|_{L^{p_0}}^{p_0} \alpha^{p-1} d\alpha \lesssim \int_0^\infty \int_\Omega \frac{1}{\alpha^{p_0}} |f^\alpha|^{p_0} \alpha^{p-1}\, d\mu\, d\alpha$

$= \int_0^\infty \int_{\{f \ge \frac{\alpha}{2}\}} \frac{1}{\alpha^{p_0}} |f|^{p_0} \alpha^{p-1}\, d\mu\, d\alpha$ Note $p > p_0$.

$\le \int_0^{2f(x)} \alpha^{p-1-p_0}\, d\alpha \approx \frac{1}{p-p_0} |f(x)|^{p-p_0}$

第二项类似. 用 p_1 即可.