Tensors are invariant under coordinate transf. but tensor components are NOT.

Let V be a vector space. $\{e_i\}_{i=1}^n$ $\{\tilde{e}_i\}_{i=1}^n$ be two basis. Einstein's summation rule applies.

Vectors. (0,1) tensor

$\tilde{e}_1 = F_1^j e_j \quad \tilde{e}_2 = F_2^i e_i \quad \cdots, \quad \tilde{e}_n = F_n^i e_i$ (or. $\tilde{e}_i = F_i^j e_j$).

$$[F] = [F_i^j] = \begin{bmatrix} F_1^1 & F_2^1 & \cdots & F_n^1 \\ F_1^2 & F_2^2 & \cdots & F_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ F_1^n & F_2^n & \cdots & F_n^n \end{bmatrix} \qquad B = F^{-1} \text{ i.e. } B_j^i F_k^j = \delta_k^i$$

a_{11}, a_{12}, a_{21}, a_{n1}

$$\underbrace{[\tilde{e}_1\ \tilde{e}_2 \cdots \tilde{e}_n]}_{\text{new}} = \underbrace{\Big[\ F.\ \Big]}_{\text{Forward transf.}} \underbrace{[e_1\ e_2, \cdots, e_n]}_{\text{old}}$$

old basis $\xrightarrow{F}$ new basis, new basis $\xrightarrow{B}$ old basis

note the reversed order

old componenet $\xrightarrow{B}$ new component, new component $\xrightarrow{F}$ old componenet

thus vectors are contra-variant tensor. vector components are contravariant.

if $v = v^i e_i = \tilde{v}^i \tilde{e}_i$, then

$$\tilde{v}^i = B_j^i v_i \quad \text{or} \quad \underbrace{\begin{bmatrix}\tilde{v}^1\\ \vdots \\ \tilde{v}^n\end{bmatrix}}_{\text{new}} = [B] \underbrace{\begin{bmatrix} v^1 \\ \vdots \\ v^n\end{bmatrix}}_{\text{old.}}$$

$\tilde{e}_i = F_i^j e_j \qquad \tilde{v}^i = B_j^i v^j$

$e_i = B_i^j \tilde{e}_j \qquad v^j = F_i^j v^i$

Covectors.

$\alpha \in V^* = \mathrm{Hom}(V, \mathbb{R})$ is called a co-vector. (1,0) tensor

$$\alpha = \sum_{i=1}^n \alpha_i e^i \quad \text{where } e^i(e_j) = \delta_j^i \Rightarrow e^i(v) = v^i \text{ where } v = v^i e_i.$$

$$\Rightarrow \alpha(e_j) = \sum \alpha_i e^i(e_j) = \sum \alpha_i \delta_j^i = \alpha_k \Rightarrow \alpha = \alpha(e_j) e^j.$$

Under coordinate change: $\alpha = \alpha_i e^i = \tilde{\alpha}_i \tilde{e}^i$. We know $\tilde{e}_i = F_i^j e_j$.

Find out $\tilde{\alpha}_i$: $\underbrace{\tilde{\alpha}_i}_{\text{new}} = \alpha(\tilde{e}_i) = \alpha(F_i^j e_j) = F_i^j \underbrace{\alpha_j}_{\text{old.}}$

$$\Rightarrow [\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n]_{\tilde{e}^i} = [\alpha_1, \cdots, \alpha_n]_{e^i} [F].$$

$\tilde{e}^i = B_j^i e^j, \quad \tilde{\alpha}_i = F_i^j \alpha_j$

$e^i = F_j^i \tilde{e}^j \qquad \alpha_i = B_i^j \tilde{\alpha}_j$

Covectors are co-variant, covector components are covariant.

Linear transf.

$L: V \longrightarrow V$. (1,1) tensor: $V^* \otimes V$.

$[L] = [L_j^i]$ means $L(e_j) = L_j^i e_i$

Under base change: $\tilde{L}_j^i \tilde{e}_i = L(\tilde{e}_j) = L(F_j^k e_k) = F_j^k L(e_k) = F_j^k L_k^p e_p$

$= F_j^k L_k^p B_p^i \tilde{e}_i$

It shows that $\tilde{L}_j^i = B_p^i L_k^p F_j^k$ i.e. $[\tilde{L}]_{\tilde{e}_i} = [B][L]_{e_i}[F]$.

$\tilde{L}_j^i = B_p^i L_k^p F_j^k$

$L_j^i = F_p^i \tilde{L}_k^p B_j^k$

$$L\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}_{e_i} \overset{L}{=} \Big(\quad\Big)_{e_i}$$

new component to old component. $\uparrow F \qquad \downarrow B$. old components to new components

$$\tilde{L}\begin{pmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{pmatrix}_{\tilde{e}_i} = \begin{bmatrix} ? \\ ? \end{bmatrix}_{\tilde{e}_i}$$

One may also express L as tensor product of $V^* \otimes V$: $L = L^i_j \, e^j \otimes e_i$

$L(e_k) = L^i_j \, e^j(e_k) \, e_i = L^i_j \, \delta^j_k \, e_i = L^i_k e_i$. as expected.

$L = L^k_l \, e^l \otimes e_k = L^k_l (F^l_j \tilde{e}^j) \otimes (B^i_k \tilde{e}_i) = \underbrace{B^i_k L^k_l F^l_j}_{\tilde{L}^i_j} \, \tilde{e}^j \otimes \tilde{e}_i$ which immediately shows $\tilde{L}^i_j = B^i_k L^k_l F^l_j$

Metric Tensor $g: V \times V \longrightarrow \mathbb{R}$.

$(0,2)$ tensor symmetric positive-defined bilinear form. $V^* \otimes V^*$.

$g_{ij} = \langle e_i, e_j \rangle \Rightarrow \langle X, Y \rangle = g_{ij} X^i Y^j = [x^1 \cdots x^n][\, g_{ij} \,]\begin{bmatrix} Y^1 \\ \vdots \\ Y^n \end{bmatrix}$. if $X = X^i e_i$ $Y = Y^j e_j$ $\Rightarrow g = g_{ij} \, e^i \otimes e^j$.

Angles: $\cos\theta = \dfrac{\langle X, Y \rangle}{\|X\| \, \|Y\|}$

$\underbrace{\tilde{e}_i}_{\text{new}} = F^j_i \underbrace{e_j}_{\text{old}}$

$e_i = B^j_i \tilde{e}_j$

Under change of Coordinate: $\tilde{g}_{ij} = \langle \tilde{e}_i, \tilde{e}_j \rangle = \langle F^k_i e_k, F^l_j e_l \rangle = F^k_i F^l_j \langle e_k, e_l \rangle = F^k_i F^l_j \, g_{kl}$.

$\Rightarrow g_{ij} = B^k_i B^l_j \tilde{g}_{kl}$

$\underbrace{\tilde{g}_{ij}}_{\text{new}} = F^k_i F^l_j \underbrace{g_{kl}}_{\text{old}}$ co-variant rules

$g_{ij} = B^k_i B^l_j \tilde{g}_{kl}$

$\|v\|^2 = \langle v, v \rangle_{\tilde{e}_i} = \underbrace{\tilde{v}^i \tilde{v}^j \tilde{g}_{ij}}_{\text{new metric}} = (B^i_a v^a)(B^j_b v^b)(F^k_i F^l_j g_{kl})$

$\tilde{v}^i = B^i_j v^j$

$v^j = F^j_i v^i$

$= \underbrace{B^i_a F^k_i}_{\delta^k_a} \underbrace{B^j_b F^l_j}_{\delta^l_b} v^a v^b g_{kl}$

$= \delta^k_a \delta^l_b v^a v^b g_{kl} = \underbrace{v^k v^l g_{kl}}_{\text{old metric}}$ as expected.

Rising and lowering indices.

$\underbrace{\tilde{g}^{ij}}_{\text{new}} = B^i_k B^j_l \underbrace{g^{kl}}_{\text{old}}$

$g^{ij} = F^i_k F^j_l \, \tilde{g}^{kl}$

in a opposite way to the change of basis thus "contravariant"

Def. $[g^{ij}] = [g_{ij}]^{-1}$. $\quad g_{ij} \, g^{jk} = g^{ij} g_{jk} = \delta^i_k = \delta^k_i$.

By Riesz representation Thm. $\forall v^* \in V^*$. $\exists!\, v \in V$. $v^*(\cdot) = \langle v, \cdot \rangle$ Hence $\overset{v_i e^i}{\overset{\parallel}{v^*}} = \langle v, \cdot \rangle = g(v, \cdot) = g_{ij} v^j e^i$.

$\Rightarrow v_i = g_{ij} v^j$. OR $v^i = g^{ij} v_j$ Warning: $v^i \neq v_i$ unless $g_{ij} = \delta^i_j$ standard Euclidean metric.

To lower the index, multiply g_{ij} To raise index, multiply g^{ij}

Eg. 1) $Q = Q^i_{jk} \, e_i \otimes e^j \otimes e^k \in V \otimes V^* \otimes V^*$. $\quad Q^i_{\cancel{j}k} \, g^{\cancel{j}x} = Q^{ix}_k \quad Q' = Q^{ix}_k \, e_i \otimes e_x \otimes e^k \in V \otimes V \otimes V^*$.

2) $D = D^{ab} e_a e_b \in V \otimes V$. $\quad D^{\cancel{a}b} g_{\cancel{a}x} = D^b_x$. $\quad D' = D^b_x \, e^x \otimes e_b \in V^* \otimes V$.

Lower both indices: $D^{ab} g_{ax} g_{by} = D_{xy} \quad D'' = D_{xy} \, e^x \otimes e^y \in V^* \otimes V^*$.

Algebraic definition. A (p,q) tensor $T: \underbrace{V\times\cdots\times V}_{p} \times \underbrace{V^*\times\cdots\times V^*}_{q} \to \mathbb{R}$ is an element of $\underbrace{V^*\otimes\cdots\otimes V^*}_{p} \otimes \underbrace{V\otimes\cdots\otimes V}_{q}$, which is multilinear. It's components $T^{i_1\cdots i_p}_{j_1\cdots j_q} = T(e_{i_1},\cdots,e_{i_p},e^{j_1},\cdots,e^{j_q})$ $\Rightarrow T = T^{i_1\cdots i_p}_{j_1,\cdots,j_q}\, e_{i_1}\otimes\cdots\otimes e_{i_p}\otimes e^{j_1}\otimes\cdots\otimes e^{j_q}$

Tensors are invariant under coordinate transf. but tensor components are NOT. Different bases give different tensor components. The transf. rules are.

$$\tilde{T}^{i_1\cdots i_p}_{j_1\cdots j_q} = B^{i_1}_{k_1}\cdots B^{i_p}_{k_p}\, T^{k_1\cdots k_p}_{l_1\cdots l_q}\, F^{l_1}_{j_1}\cdots F^{l_q}_{j_q}$$

Christoffel Symbols.

Recall surfaces in $\mathbb{R}^3$. Consider a curve $\gamma(s)$ $\overset{\parallel}{}$ $F(u(s),v(s))$, where $F(u_1,u_2)$ is a local chart. (para- by arc-length) on S. The acceleration

$$\gamma'(s) = \frac{\partial F}{\partial u}\frac{du}{ds} + \frac{\partial F}{\partial v}\frac{dv}{ds} = \frac{du_i}{ds}\frac{\partial F}{\partial u_i} \quad (i=1,2.\ u_1=u.\ u_2=v).$$

$$\gamma''(s) = \frac{d}{ds}\left(\frac{du_i}{ds}\frac{\partial F}{\partial u_i}\right) = \frac{d}{ds}\left(\frac{\partial F}{\partial u_i}\right)\frac{du_i}{ds} + \frac{\partial F}{\partial u_i}\frac{d^2u_i}{ds^2} = \underbrace{\frac{\partial^2 F}{\partial u_i\partial u_j}}_{\text{has tengent component and normal component.}}\frac{du_i}{ds}\frac{du_j}{ds} + \underbrace{\frac{\partial F}{\partial u_i}}_{\text{tangent to the surface.}}\frac{d^2u_i}{ds^2}$$

Def. Γ^k_{ij} as: $\dfrac{\partial^2 F}{\partial u_i\partial u_j} = \underbrace{\Gamma^k_{ij}\dfrac{\partial F}{\partial u_k}}_{\text{tangential component.}} + \underbrace{L_{ij}}_{\text{II-form } [\text{II}] = [L_{ij}]}\vec{n}$.

then $\gamma''(s) = \left(\Gamma^k_{ij}\dfrac{du_i}{ds}\dfrac{du_j}{ds} + \dfrac{d^2u_k}{ds^2}\right)\dfrac{\partial F}{\partial u_k} + L_{ij}\dfrac{du_i}{ds}\dfrac{du_j}{ds}\vec{n}$

Find out Γ^k_{ij}: $\dfrac{\partial^2 F}{\partial u_i\partial u_j}\cdot\dfrac{\partial F}{\partial u_l} = \Gamma^k_{ij}\dfrac{\partial F}{\partial u_k}\cdot\dfrac{\partial F}{\partial u_l} = \Gamma^k_{ij}\, g_{kl}$.

$$\Rightarrow \Gamma^k_{ij} = \left(\frac{\partial^2 F}{\partial u_i\partial u_j}\cdot\frac{\partial F}{\partial u_l}\right) g^{lk} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l}\right) \quad (*)$$

Geodesic Eq. tangential component of $\gamma''(s) \equiv 0$

$$\Rightarrow \Gamma^k_{ij}\frac{du_i}{ds}\frac{du_j}{ds} + \frac{d^2u_k}{ds^2} = 0 \qquad k=1,2.$$

proof of $(*)$: $\dfrac{\partial g_{ij}}{\partial u_l} = \dfrac{\partial}{\partial u_l}\left\langle \dfrac{\partial F}{\partial u_i}, \dfrac{\partial F}{\partial u_j}\right\rangle = \left\langle \dfrac{\partial^2 F}{\partial u_l\partial u_i}, \dfrac{\partial F}{\partial u_j}\right\rangle + \left\langle \dfrac{\partial F}{\partial u_i}, \dfrac{\partial^2 F}{\partial u_j\partial u_l}\right\rangle$.

Since $\dfrac{\partial F}{\partial u_i\partial u_j} = \Gamma^k_{ij}\dfrac{\partial F}{\partial u_k} + L_{ij}\vec{n}$. we have.

$$\frac{\partial g_{ij}}{\partial u_l} = \left\langle \Gamma^p_{il}\frac{\partial F}{\partial u_p} + \underbrace{L_{il}\vec{n}}_{\text{normal}}, \frac{\partial F}{\partial u_j}\right\rangle + \left\langle \frac{\partial F}{\partial u_i}, \Gamma^q_{jl}\frac{\partial F}{\partial u_q} + \underbrace{L_{jl}\vec{n}}_{\text{normal}}\right\rangle$$

$$= \Gamma^p_{il}\, g_{pj} + \Gamma^p_{jl}\, g_{ip}.$$

Note. $g_{ij} = g_{ji}$

$\Gamma^k_{ij} = \Gamma^k_{ji}$

Christoffel Symbol of the 2nd kind.

- - - - - - - first kind: $\Gamma_{kij} = g_{kl}\Gamma^l_{ij}$.

Similarly, $\dfrac{\partial g_{il}}{\partial u_j} = \Gamma^p_{ij}\, g_{pl} + \Gamma^p_{jl}\, g_{pi}$

Geometric meaning: it tracks how basis vector change from point to point.

$$\frac{\partial g_{jl}}{\partial u_i} = \Gamma^p_{ij}\, g_{pl} + \Gamma^p_{il}\, g_{pj}$$ The theorem follows.

Properties: • $\Gamma^k_{ij} = \Gamma^k_{ji}$, follows from mixed partial derivative thm.

• $\Gamma_{kij} = \Gamma_{kji}$.

References:

- Lecture Notes on Diff. Mnflds and Rie- Geo By Prof Fong. Tsz Ho.
- Lecture Notes on Diff. Geo. By Prof. Peter Petersen
- Youtube video on "Tensors for beginners" and "Tensor Calculus".