Sobolev Space Ref. Evans PDE Chapter 5.

Def. Hölder Space. Let $U \subseteq \mathbb{R}^n$ be open.

(1) If $u: U \to \mathbb{R}$ bounded cts. We write $\|u\|_{C(\overline{U})} := \sup_{x\in U} |u(x)|$

(2) $0 < \gamma \leq 1$. γ-th Hölder seminorm of u is given by

$$[u]_{C^{0,\gamma}(\overline{U})} := \sup_{\substack{x,y\in U \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}.$$

Hölder norm $\|u\|_{C^{0,\gamma}(\overline{U})} := \|u\|_{C(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}$

RMK. • Other convention: $[u]_{C^{0,\gamma}}$, $[u]_{\dot{C}^{\gamma}}$

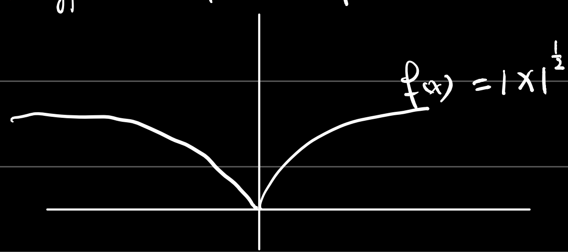
• $u \in C(U)$ often means u is cts, without the norm.

Def $k \geq 1$ integer. $C^{k,\gamma}(\overline{U}) := \{u \in C^k(U)\}$ has norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})}$$

where $\alpha = \alpha_1 + \cdots + \alpha_n$ $D^\alpha = \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ multi-index.

RMK. Typical C^γ-function $f(x) = |x|^\gamma$, $\gamma = \frac{1}{2}$.



Smooth functions are NOT dense in C^γ-space!

RMK. $B_t(W_t): [0,\infty) \to \mathbb{R}$ Brownian Motion. typical path is Hölder cts with $0 < \gamma < \frac{1}{2}$.

Thm. $C^{k,\gamma}$ is Banach Space.

Proof. Exercise.

RMK. Polynomial Approximation $\inf_{\deg \leq k} \|f - P(x)\|_{C^\gamma}$

Sobolev Spaces.

Motivation: Hölder Spaces good. but not good for PDE estimates. Many sols are not Hölder!

Def. Weak derivative. $U \subseteq \mathbb{R}^n$ open

$C_c^\infty(U) := \{ u \in C^\infty(U) \text{ with compact supp. in } U \}$.

$\phi \in C_c^\infty(U)$ a test function.

The Weak derivative of f is such a function h. s.t. $\longleftarrow$ 1D version.

$$\underset{U}{\int} \underbrace{f}_{\in L^1_{loc}} \phi' \, dx = - \underset{U}{\int} \phi \underbrace{h}_{\in L^1} \, dx \quad \text{(integration by parts).} \quad \forall \phi \in C_c^\infty(U).$$

sufficient. $\longrightarrow$ $u \in L^1_{loc}(U) := \{ \int_K |u| < +\infty \ \forall K \subseteq U \text{ compact} \}$.

Classically if $u \in C^k(U)$:

$$\int_U u \, D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U (D^\alpha u) \cdot \phi \, dx$$

This motivates the following definition.

Def. weak derivative.

Suppose $u, v \in L^1_{loc}(U)$ $\alpha = (\alpha_1, \cdots, \alpha_n)$ multi-index.

We say v is the weak α-th derivative of u $\left(D^\alpha u \overset{L^1_{loc}(U)}{=\!=} v. \right)$ if

Δ unique up to a zero measure set.

$$\int_U u \, D^\alpha \phi = (-1)^{|\alpha|} \int_U v \phi \quad \forall \phi \in C_c^\infty(U).$$

RMK. v is unique in L^1_{loc} sense: if v_1, v_2 are α-th weak derivative of u, then $\int_K v_1 = \int_K v_2$ $\forall K$ compact $\subseteq U$.

i.e. $v_1 = v_2$ except a measure zero set.

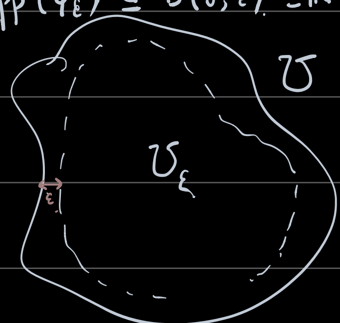
Example. See Evans Chapter 5 Section 2, Example 1, 2.

Mollifier 磨光子. $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi\left(\frac{x}{\varepsilon}\right)$ $\operatorname{supp}(\phi_\varepsilon) \subseteq B(0, \varepsilon) \subseteq \mathbb{R}^n$

1) $\phi \in C_c^\infty(\mathbb{R}^n)$

2) $\int_{\mathbb{R}^n} \phi(x) \, dx = 1 = \int_{\mathbb{R}^n} \phi_\varepsilon(x) dx$.

3) $\lim_{\varepsilon \to 0} \phi_\varepsilon(x) = \delta(x)$.



Suppose $f: U \to \mathbb{R}. \in L^1_{loc}(U)$. Define mollification of f $\quad f^\varepsilon := \phi_\varepsilon * f$.

$$f^\varepsilon(x) = \int_{B(x,\varepsilon)} \phi_\varepsilon(x-y) f(y) dy = \int_{B(0,\varepsilon)} \phi_\varepsilon(y) f(x-y) dy \quad \forall x \in U_\varepsilon := \{ x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon \}$$

Thm. Properties of mollifiers   mollifier $\subsetneq$ approximation to the identity

1) $f^\varepsilon \in C^\infty(U_\varepsilon)$ since convolution is a smoothing operator.

2) $f^\varepsilon \to f$ a.e. $x \in U$ as $\varepsilon \to 0$

Proof. $$|f^\varepsilon(x) - f(x)| = \left| \int_{B(x,\varepsilon)} \phi_\varepsilon(x-y) f(y)\,dy - \int_{B(x,\varepsilon)} f(x) \underbrace{\phi_\varepsilon(x-y)\,dy}_{\int \phi_\varepsilon = 1} \right|$$

$$\le \int_{B(x,\varepsilon)} |\phi_\varepsilon(x-y)|\,|f(x) - f(y)|\,dy$$

$$\lesssim \int_{B(x,\varepsilon)} |f(y) - f(x)|\,dy \longrightarrow 0$$ if x is a Lebesgue point, by Lebesgue Diff. Thm.

RMK. Consequence of Lebesgue Diff. Thm. and 2): proof of uniqueness of weak derivative.

3) if $f \in C(U)$. Then $f^\varepsilon \rightrightarrows f$ $\forall$ compact $K \subseteq U$.

Proof. Almost the same as above.

4) If $1 \le p < +\infty$. Assume $f \in L^p_{loc}(U)$ then $f^\varepsilon \to f$ in $L^p_{loc}(U)$

RMK.

$$\lim_{\varepsilon \to 0} \sup_{|y| \le 1} \| \chi_E(x - \varepsilon y) - \chi_E(x) \|_{L^p(\mathbb{R}^n)} = 0$$

Hint: ① Regularity of Lebesgue measure. approximate E using compact $K \subseteq E$

② enlarge K a little by compactness

Summary $u \in L^1_{loc}(U)$ if $v \in L^1_{loc}(U)$ satisfies

- Weak derivative. $\int v\,\phi = (-1)^{|\alpha|} \int D^\alpha\phi \cdot u$

Then $D^\alpha u := v$

RMK (exercise) $f: [0,1] \mapsto [0,1]$ be the Cantor function. Does f has a weak derivative?

Defition of Sobolev Space $1 \le p \le +\infty$, $k \in \mathbb{N}$.

$$W^{k,p}(U) = \{ u \in L^1_{loc}(U) : \forall\, |\alpha| \le k,\ D^\alpha u \text{ exists in the weak sense and } D^\alpha u \in L^p(U) \}$$

(k ← regularity, p ← integrability)

RMK. 1) $p=2$ write $H^k(U) := W^{k,2}$. "H" means Hilbert. $H^0 = L^2(U)$

2) identify $u = v$ if $v = u$ a.e.

Def. (Norm of $W^{k,p}$)

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}} \\ \sum_{|\alpha| \le k} \operatorname{ess\,sup} \|D^\alpha u\| \end{cases}$$

$\downarrow$

Equivalently $\sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p}$

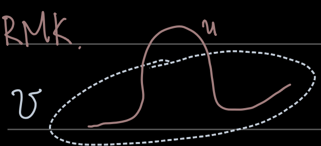
Notion of convergence: $u_m \to u$ in $W^{k,p}_{loc}(U)$

$\Leftrightarrow u_m \to u$ in $W^{k,p}(V)$ for each $V \subset\subset U$.

Def. $W^{k,p}_0(U) :=$ closure of $C^\infty_c(U)$ in $\|\cdot\|_{W^{k,p}(U)}$ $u \in W^{k,p}_0$ iff $\exists \{u_m\}_{m=1}^\infty \in C^\infty_c$, $u_m \to u$ in $W^{k,p}(U)$

$H^k := W^{k,2}$, a Hilbert Space. $H^k_0 = W^{k,2}_0$ eg. $H^0 = W^{0,2} = L^2(U)$.

RMK.



$u \in W^{k,p}_0(U)$

$\Leftrightarrow \begin{cases} u \in W^{k,p}(U) \\ D^\alpha u = 0 \text{ on } \partial U,\ \forall\, |\alpha| \le k-1 \end{cases}$ (for $k \ge 1$)

RMK. In 1D. $U = (a,b)$. $u \in W^{1,p}(U) \Leftrightarrow u \overset{a.e.}{=\!=}$ an absolute cts function v s.t. $v' \in L^p$.

Eg $U = B(0,1) = \{x \in \mathbb{R}^n \mid |x| \le 1\}$. Take $u(x) = |x|^{-\alpha}$ $\forall x \in U \setminus \{0\}$

Claim: $u \in W^{1,p}(U)$ iff. $\alpha < \frac{n-p}{p}$

$u \notin W^{1,p}(U)$ if $p \ge n$.

Proof: Guess a weak derivative: ordinary derivative $\frac{\partial u}{\partial x_i} = \frac{-\alpha x_i}{|x|^{\alpha+2}}$ $\forall x \in U \setminus \{0\}$ smooth away from 0.

WANT to check: $\int_U \partial_i u \cdot \phi \overset{?}{=} -\int u\, \partial_i \phi$ $\forall \phi \in C^\infty_c(U)$.

Singularity at zero: natural strategy: take $B(0,\varepsilon)$ away from U, and then send $\varepsilon \to 0^+$.

$$\forall \varepsilon>0: \quad \int_{U-B(0,\varepsilon)} u \cdot \partial_i \phi = -\int_{U-B(0,\varepsilon)} \partial_i u \cdot \phi + \int_{\partial B(0,\varepsilon)} u\phi \cdot \nu_i \, d\sigma$$

← area measure

in-ward pointing unit normal $\nu = (\nu_1, \dots, \nu_n)$

Green's identity (integration by parts)

Estimate the boundary term: $\left| \int_{\partial B(0,\varepsilon)} u \phi \nu_i \, d\sigma \right| \leq \|\phi\|_\infty \int_{\partial B(0,\varepsilon)} \varepsilon^{-\alpha} d\sigma \lesssim \varepsilon^{n-\alpha-1} \xrightarrow{\varepsilon \to 0} 0$ if $\alpha < n-1$.

Satisfied.

$$\underbrace{|Du|}_{\nabla u} \lesssim |x|^{-(\alpha+1)} \quad \nabla u \in L^p \Leftrightarrow (\alpha+1)p < n \Leftrightarrow \alpha < \frac{n-p}{p}$$

Eg. ($W^{1,p}$ functions can behave badly) not bounded, like $u(x) = |x|^{-\alpha}$ ($\alpha < \frac{n-p}{p}$)

Take $\{r_k\}_{k=1}^\infty$ be rational pts in $U = B(0,1)$ (or any countable dense subset)

Define $u(x) := \sum_{k=1}^\infty 2^{-k} |x - r_k|^{-\alpha}$. $x \in U$. Check $u \in W^{1,p}(U)$ if $\alpha < \frac{n-p}{p}$

$0 < \alpha < \frac{n-p}{p} \Rightarrow u(x)$ is unbounded on any open subset of U.

Thm. Elementary properties of weak derivative. Assume $u, v \in W^{k,p}(U)$ $|\alpha| \leq k$.

- linearity $D^\alpha (Au + Bv) = A D^\alpha u + B D^\alpha v$
- $D^\alpha u \in W^{k-\alpha, p}$
- Mixed derivative: $D^\alpha D^\beta u = D^\beta (D^\alpha u) = D^{\alpha+\beta} u \quad \forall\, |\alpha| + |\beta| \leq k$.

Check by def. Fix $\phi \in C_c^\infty(U)$. $\int_U D^\alpha u \underbrace{D^\beta \phi}_{\in C_c^\infty} = (-1)^{|\alpha|} \int_U D^{\alpha+\beta}\phi \cdot u = (-1)^{|\alpha|} (-1)^{|\alpha|+|\beta|} \int_U D^\alpha u \cdot \phi$

$= (-1)^{|\beta|} \int_U D^\alpha u \cdot \phi$

- Libniz rule. $\forall \phi \in C_c^\infty$, $u \in W^{k,p}$: $\phi u \in W^{k,p}$. $D^\alpha(\phi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u \, D^{\alpha-\beta} \phi$

Proof by induction.

multi-binomial thm.

$$\binom{\alpha}{\beta} = \prod_{j=1}^n \frac{(\alpha_j)!}{\beta_j! \, (\alpha_j - \beta_j)!}$$

Thm. Sobolev Spaces are Banach.

- $k=0$: $W^{0,p} = L^p$ Banach
- For $k = 1, 2, 3, \dots$, $1 \le p \le \infty$: ↙ Evans. Theorem 2 Chapter 5.

Proof: 1) $\|\cdot\|_{W^{k,p}}$ is a norm.

2) Completeness Consider a Cauchy Sequence $\{u_m\}$ in $W^{k,p}$.

For each $|\alpha| \le k$: $\{D^\alpha u_m\}$ is Cauchy in L^p. By Completeness of L^p, $\exists\, u_\alpha \in L^p$ $D^\alpha u_m \to u_\alpha$.

In particular, $\exists\, u_{(0,0,\dots,0)}$ s.t. $u_m \longrightarrow u_{(0,0,\dots,0)} := u \in L^p$.

3) Claim: $u := u_{(0,0,\dots,0)} \in W^{k,p}$ and $D^\alpha u = u_\alpha$.

To prove the claim, take $\phi \in C_c^\infty(U)$. $\int_U u\, D^\alpha \phi = \int \lim_{m\to\infty} u_m \cdot D^\alpha \phi \overset{\text{LDCT}}{=} \lim_{m\to\infty} \int u_m D^\alpha \phi$

$$= \lim_{m\to\infty} (-1)^{|\alpha|} \int D^\alpha u_m \cdot \phi$$

$$\overset{\text{LDCT}}{=\!=} (-1)^{|\alpha|} \int u_\alpha \cdot \phi$$

$\Rightarrow D^\alpha u = u_\alpha$.

Since $\exists\, D^\alpha u \in L^p$ $D^\alpha u_m \to D^\alpha u$, $\forall\, |\alpha| \le k$ we have $u \in W^{k,p}$. ∎

Approximation of Sobolev functions week 2.

- Interior approximation by smooth functions

Thm. (Local approximation) Assume $u \in W^{k,p}$, $u \in L^1_{loc}$ for some $1 \le p < \infty$.

set $u^\varepsilon = \underbrace{\eta_\varepsilon}_{\text{mollifier}} * u$ in $U_\varepsilon := \{ x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon \}$ Then:

(i) $u^\varepsilon \in C^\infty(U_\varepsilon)$ $\forall\, \varepsilon > 0$

(ii) $u^\varepsilon \longrightarrow u$ in $W^{k,p}_{loc}(U)$ as $\varepsilon \to 0$. Warning: not true if $p = +\infty$.

Proof. (ii). Claim: $\underbrace{D^\alpha u^\varepsilon}_{\text{ordinary derivative}} = \eta_\varepsilon * D^\alpha u$

Proof of the claim: Fix $x \in U_\varepsilon$. $(D^\alpha u^\varepsilon)(x) = D^\alpha\left[\int_U u(y)\, \eta_\varepsilon(x-y)\, dy\right] \overset{\text{proved before}}{=} \int_U \underbrace{(D_x^\alpha \eta_\varepsilon)}_{\in C_c^\infty(U)}(x-y)\, u(y)\, dy = (-1)^{|\alpha|} \int_U (D_y^\alpha \eta_\varepsilon)(x-y) \cdot u(y)\, dy$

$\overset{\text{def of weak}}{=} (-1)^{|\alpha|+|\alpha|} \int \eta_\varepsilon(x-y)\, D^\alpha u(y)\, dy$

derivative

$$= \int_U \eta_\varepsilon(x-y)\cdot D^\alpha u(y)\,dy = \eta_\varepsilon * D^\alpha u.$$

Proof of Convergence in $W^{k,p}_{loc}$.

Pick any $V \subset\subset U$. choose ε small enough s.t. $V \subset U_\varepsilon$.

WANT: $\|u-u^\varepsilon\|_{W^{k,p}(V)} \to 0$

$$\|u-u^\varepsilon\|^p_{W^{k,p}(V)} := \sum_{|\alpha|\le k} \|D^\alpha u^\varepsilon - D^\alpha u\|^p_{L^p(V)}$$

$$= \sum_{|\alpha|\le k} \|D^\alpha u - \eta_\varepsilon * D^\alpha u\|^p_{L^p(V)} \longrightarrow 0 \text{ by property of mollifier.}$$

RMK. Convergence fails if $p=+\infty$ (Exercise): $U=(-1,1)$ $u(x)=|x|$. $u'(x)=\frac{x}{|x|}$ $(x\neq 0)$ $u\in W^{1,\infty}(U)$ but u cannot be approximated by C^∞-functions under $\|\cdot\|_{W^{1,\infty}}$

• Global Approximation by smooth functions.

Thm. Assume U open, bounded. Take $u\in W^{k,p}(U)$ where $1\le p<+\infty$. Then:

$\exists$ functions $\{u_m\}_{m=1}^\infty \in C^\infty(U)\cap W^{k,p}(U)$ s.t. $u_m \to u$ in $W^{k,p}(U)$

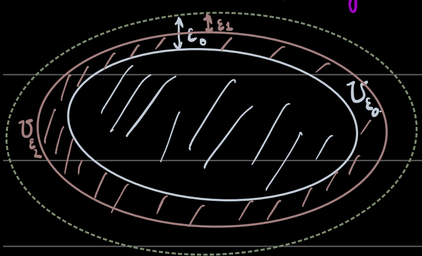
RMK.

1) No condition on the boundary ∂U

2) only have $\boxed{u_m \in C^\infty(U)}$ other than $C^\infty(\bar U)$

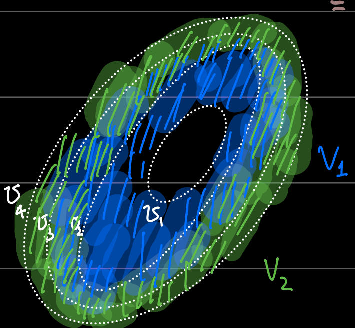
in particular can be not smooth on ∂U.

Proof. Idea: Partition of unity and mollifier.



Recall Partition of Unity. $U=\bigcup_{i=1}^\infty U_i$ where $U_i=\{x\in U \mid dist(x,\partial U)>\frac{1}{i}\}$

$$= \bigcup_{i=1}^\infty V_i \quad \text{where } V_i = U_{i+3} - \bar U_{i+1}$$



$\exists \{f_i\}_{i=1}^\infty$. $f_i\in C_c^\infty(V_i)$ $0\le f_i\le 1$ (locally finite)

$\forall x\in U$: $\sum_{i=1}^\infty f_i(x)=1$.

$u\in W^{k,p}(U)$: $u=\sum_{i=1}^\infty \boxed{u\cdot f_i}$ $\in C_c^\infty(V_i)$ "away from the boundary by $\frac{1}{i}$"

We could find $\eta_{\varepsilon_i}\in C_c^\infty(U)$ s.t. $\|\eta_{\varepsilon_i}*(uf_i)-uf_i\|_{W^{k,p}} \le \frac{\varepsilon}{2^{i+1}}$

Set $v=\sum^\infty \eta * (uf_i) \in C^\infty$.

$$\|v-u\|_{W^{k,p}} = \Big\|\sum \eta_{\varepsilon_i} * (uf_i) - \sum uf_i\Big\|_{W^{k,p}} \le \sum \|\quad\|_{W^{k,p}} = \sum \varepsilon \cdot 2^{-(i+1)} = \varepsilon.$$

Problem: To avoid infinite summation, Take $V \subset\subset U$

$\Rightarrow v = \sum \eta_{\varepsilon_i} * (uf_i)$ is a finite summation on $\overline{V}$. $\|v-u\|_{W^{k,p}(V)} \le \varepsilon$.

Take sup over all V. then $\|u-v\|_{W^{k,p}(U)} \le \varepsilon$.

• Global Approximation including ∂U.

Thm. Approximation by smooth functions up to boundary.

Assume U is bounded. ∂U is C^1.

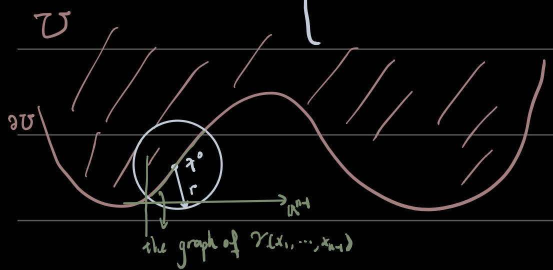
Suppose $u \in W^{k,p}(U)$ with $1 \le p < +\infty$ Then $\exists \{u_m\}_{m=1}^{\infty} \in C^\infty(\overline{U})$ s.t. $u_m \to u$ in $W^{k,p}(U)$

RMK Lipschitz boundary is fine.

Def. Regularity of boundary.

Let $U \subseteq \mathbb{R}^n$ be open and bounded. $k \in \mathbb{N}$. We say ∂U is C^k if $\forall x^0 \in \partial U$. one can find $r > 0$. and C^k function $\gamma : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ s.t. (up to switching coordinates if necessary)

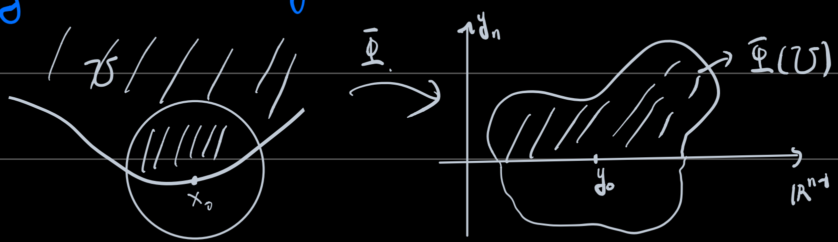
$$U \cap B(x^0, r) = \left\{ x \in B(x^0, r) \,\middle|\, x_n > \gamma(x_1, \cdots, x_{n-1}) \right\}$$ similar with def in manifold theory



Def. If ∂U is C^1, then along the boundary we could define the outward unit normal vector field $\nu = (\nu_1, \cdots, \nu_n)$

If $u \in C^\infty(\overline{U})$ we define $\frac{\partial u}{\partial \nu} := Du \cdot \nu$

Straightening the boundary



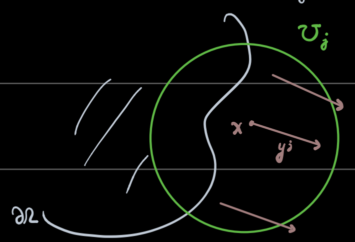
$$\begin{cases} y_i = x_i & (i=1,\cdots,n-1) \\ y_n = x_n - \gamma(x_1,\cdots,x_{n-1}) \end{cases}$$

Question: under what condition can we approximate $W^{k,p}(U)$ by $C^\infty(\bar{U})$?

Segment Condition.

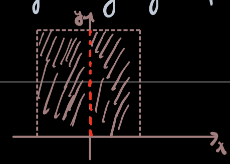
We say a bounded open set $\Omega \subseteq \mathbb{R}^n$ satisfies the "segment condition" if there is an open covering of $\bar{\Omega}$: $U_0, \cdots, U_N$.

s.t. • $U_0 \subseteq \Omega$ • $U_j \cap \partial\Omega \neq \phi$ $1 \le j \le N$. • $\forall j \ge 1$, $\exists\, y^j \in \mathbb{R}^n$. $\forall x \in U_j \setminus \Omega$, $\forall\, 0 < \varepsilon \le 1$. $x + \varepsilon y^j \notin \bar{\Omega}$



"If you are outside, there is a safe direction that you may go further".

Counter-example: $\Omega = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} 0 < |x| < 1 \\ 0 < y < 1 \end{array} \right\}$



RMK. If ∂U is C^1 or Lipschitz, then the segment condition is true.

Thm. If $U \subseteq \mathbb{R}^n$ open satisfies the segment condition, then the ~~set of restriction to U of $C_c^\infty(\mathbb{R}^n)$~~ almost $C^\infty(\bar{U})$ is dense in $W^{k,p}(U)$. for $1 \le p < +\infty$

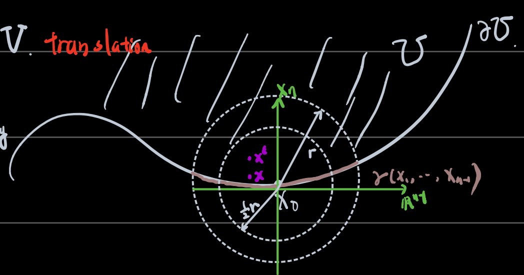
Proof. Special Case: U bounded. ∂U is C^1.

Idea: 1) Smooth partition of unity. 2) shifting trick.

Fix $x_0 \in \partial U$. Since ∂U is C^1. WLOG. assume $U \cap B(x_0, r) = \{ x \in B(x_0,r) \mid x_n > \gamma(x_1,\cdots,x_{n-1}) \}$ C^1 function.

Set $V := B(x_0, \frac{1}{2}r) \cap U$. $x^\varepsilon := x + \lambda\varepsilon e_n$. where λ is large const. ε small const. s.t. $B(x^\varepsilon, \varepsilon) \subseteq B(x_0, r) \cap U$ $\forall x \in V$

$u_\varepsilon(x) := u(x^\varepsilon) = u(x + \lambda\varepsilon e_n)$ $x \in V$. translation



Now there is sufficient room to mollify

$v^\varepsilon := \eta_\varepsilon * u_\varepsilon$

• Claim: as $\varepsilon \to 0^+$, $\|v^\varepsilon - u\|_{W^{k,p}(V)} \longrightarrow 0$ (local).

$\forall\, |\alpha| \le k$. $\|D^\alpha v^\varepsilon - D^\alpha u\|_{L^p(V)} \le \underbrace{\|D^\alpha v^\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)}}_{\substack{\text{mollification error,} \\ \text{proved } \to 0}} + \underbrace{\|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p(V)}}_{\substack{\text{shifting, won't change } L^p \text{ norm much since translation} \\ \text{is cts.}}}$

$\to 0.$

- Now partition of unity. For any $\delta > 0$: Since the boundary is compact, one can find

x_i, $1 \le i \le N$.

$V_i = U \cap B(x_i, \frac{r_i}{2})$

$v_i \in C^\infty(V_i)$ s.t. $\begin{cases} \| v^i - u \|_{W^{k,p}(V_i)} \le \delta \\ \partial U \subseteq \bigcup_{i=1}^{N} B(x_i, \frac{r_i}{2}) \end{cases}$

Also choose $V_0 \subset\subset U$, $v_0 \in C^\infty(V_0)$, s.t.

$$\| v_0 - u \|_{W^{k,p}(V_0)} \le \delta$$
$$U \subseteq \bigcup_{i=0}^{N} V_i.$$

Now let $\{\varphi_i(x)\}$ be a smooth partition of unity subordinate of $\{V_i\}_{i=0}^{N}$

Define $v(x) = \sum_{i=0}^{N} \varphi_i(x) v_i(x)$. $\in C^\infty(\overline{U})$

$$\| D^\alpha v - D^\alpha u \|_{L^p(U)} = \| \sum (D^\alpha(\varphi_i v_i) - D^\alpha(\varphi_i u)) \| \le \sum \| D^\alpha [\varphi_i (v_i - u)] \|_{L^p(U)}$$
$$\le C \cdot \sum \| u - v_i \|_{W^{k,p}(V_i)} \le C(N+1)\delta.$$

Now δ can be arbitrarily small done.

- General Case.

$f^\varepsilon(x) = f(x - \varepsilon y^i)$ Exercise.

Corollary. $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. i.e. $C_c^\infty(\mathbb{R}^n)$ dense in $W^{k,p}(\mathbb{R}^n)$ $1 \le p < +\infty$

• Chain Rule for weak derivatives

U open bounded $\subseteq \mathbb{R}^n$. We call $u \in W^k(U)$ if u has weak derivatives up to $|\alpha| = k$.

i.e. $\begin{cases} D^\alpha u \in L^1_{loc}(U) \quad \forall\ |\alpha| \le k. \\ u \in L^1_{loc}(U) \end{cases}$ Note that $C^k \subseteq W^k$.

Prop. (Chain rule - simplest case)

Assume $f \in C^1(\mathbb{R})$ $f' \in L^\infty(\mathbb{R})$ $u \in W^1(U)$

Then $f \circ u \in W^1(U)$. $D(f \circ u) = f'(u) \cdot Du$

RMK. (Exercise) Suppose $u, v \in L^1_{loc}(U)$. Then $v = D^\alpha u$ iff $\exists \{u_m\}_{n=1}^\infty \in C^\infty(U)$, $\begin{cases} u_m \to u \text{ in } L^1_{loc}(U) \\ D^\alpha u_m \to v \text{ in } L^1_{loc}(U) \end{cases}$
(Lemma)

Proof of Prop: Let $\{u_m\} \in C^1(U)$ s.t. $u_m \to u$, $Du_m \to Du$ in L^1_{loc} (by the above lemma)

Then $\forall$ compact-contained $V \subseteq U$, $\int_V |f(u_m) - f(u)| \le \|f'\|_\infty \int_V |u_m - u| \longrightarrow 0$ as $m \to \infty$.

Also, $\int_V |f'(u_m) Du_m - f'(u) Du| \le \int_V |f'(u_m) Du_m - f'(u_m) Du| + \int_V |f'(u_m) Du - f'(u) Du|$

$$\le \|f'\|_\infty \underbrace{\int_V |Du_m - Du|}_{(I)} + \underbrace{\int_V |f'(u_m) - f'(u)| \cdot |Du|}_{(II)}$$

$\to 0$ by Lemma. (II) not that trivial.

$\underbrace{\int_V |f'(u_m) - f'(u)| \cdot |Du|}_{(II)}$ We know $u_m \to u \in L^1(V)$. Hence $\exists\ u_{m_j} \to u$ a.e. pointwise in V.

with f' cts, $f'(u_{m_j}) \to f'(u)$ a.e. in V.

$\|f'\|_\infty < +\infty$. $Du \in L^1(V)$. Apply LDCT. $\int_V |f'(u_{m_j}) - f'(u)|\,|Du| \longrightarrow 0$ as $j \to \infty$.

$\blacksquare$

Prop. Let $u \in W^1(U)$. Then $u^+, u^- \in W^1$. so is $|u| = u^+ + u^-$.

$Du^+ = \begin{cases} Du & \text{if } u \ge 0 \\ 0 & \text{if } u \le 0 \end{cases}$ $Du^- = \begin{cases} -Du & \text{if } u \le 0. \\ 0 & \text{if } u > 0. \end{cases}$ $D|u| = \begin{cases} Du & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -Du & \text{if } u < 0 \end{cases}$

Proof. $\varepsilon > 0$. $\boxed{f_\varepsilon(u) = \sqrt{u^2 + \varepsilon^2}} \xrightarrow{\varepsilon \to 0} |u|$ $\in C^1(\mathbb{R})$. f_ε' is bounded (by 1) $f \in W^1$.

By previous prop. $D(f_\varepsilon(u)) = \frac{u}{\sqrt{u^2+\varepsilon^2}} Du$.

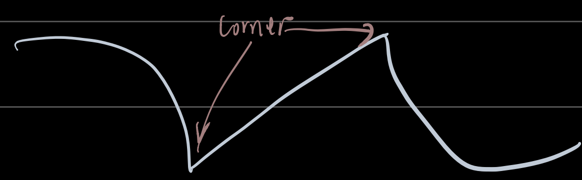
Take $\varphi \in C_c^\infty$.

$$\int f_\varepsilon(u)\, D\phi = -\int \frac{u}{\sqrt{u^2+\varepsilon^2}} Du \cdot \varphi \xrightarrow{\varepsilon\to 0} \begin{cases} 1 & \text{if } u>0 \\ 0 & \text{if } u=0 \\ -1 & \text{if } u<0 \end{cases} \quad \text{as desired.}$$

Similarly for Du^+, use $f_\varepsilon(u) = \begin{cases} (u^2+\varepsilon^2)^{\frac{1}{2}} - \varepsilon & \text{if } u \geq 0 \\ 0 & \text{if } u<0. \end{cases}$

Thm. Suppose f is a piece-wise smooth function on $\mathbb{R}$. $f' \in L^\infty(\mathbb{R})$, $u \in W^1$.

Then $f \circ u \in W^1$. $D(f\circ u) = \begin{cases} f'(u)\, Du & \text{if } u \notin \text{corner pts.} \\ 0 & \text{if } u \in \text{corner pts.} \end{cases}$



Coordinate transformation.

Def. Suppose $\Phi : \Omega \subseteq \mathbb{R}^n \mapsto \tilde{\Omega} \subseteq \mathbb{R}^n$ is bijective. We say Φ is <u>m-smooth</u> if $\Phi \in C^m(\Omega)$. $\Phi^{-1} \in C(\tilde{\Omega})$.

Call **diffeomorphism** if $\Phi, \Phi^{-1} \in C^\infty$.

Suppose u is a measurable function on Ω. Define $(Au)(y) = u(\Phi^{-1}(y))$, $y \in \tilde{\Omega}$.

Claim: $c' \| u \|_{p,\Omega} \leq \| Au \|_{p,\tilde{\Omega}} \leq c \| u \|_{p,\Omega}$ where $\| \cdot \|_{p,\Omega} = \left(\int_\Omega |\cdot|^p dx \right)^{\frac{1}{p}}$, provided $\Phi \in C^1$.

Thm. Suppose Φ m-smooth. Then $A : W^{m,p}(U) \longrightarrow W^{m,p}(\tilde{U})$ is bijective, bounded, with bounded inverse.

$$\| Au \|_{W^{m,p}(\tilde{U})} \sim \| u \|_{W^{m,p}(U)} \quad \text{where } (Au)(y) := u \circ \Phi^{-1}(y).$$

Proof: Goal: $\| Au \|_{m,p,\tilde{U}} \lesssim \| u \|_{m,p,U}$. The reverse ineq. can be shown using A^{-1}.

Idea: use C^∞ approximation. $u \in W^{m,p}(U) \Leftrightarrow \{u_j\}_{j=1}^\infty \subseteq C^\infty(U)$. $u_j \longrightarrow u$ in $W^{k,p}(U)$ (Global approximation Thm).

By induction on multi-index $|\alpha|$, it sufficient to show

$Au_j = u_j \circ \Phi^{-1}$. $D^\alpha(Au_j) = \sum (D^\beta u_j) \circ \Phi^{-1} \cdot D^{\alpha-\beta} \Phi^{-1}$

$$D^\alpha (Au_j)(y) = \sum_{\beta \leq \alpha} M_{\alpha\beta}(y) \cdot A(D^\beta u_j)(y)$$

and $M_{\alpha\beta}$ is a polynomial in derivatives of Φ^{-1} of degree $\leq \beta$

Take a test function $\theta \in C_c^\infty(\tilde{U})$.

$$(-1)^{|\alpha|} \int_{\tilde{U}} (Au_j)(y)\, (D^\alpha \theta)(y)\, dy = \sum_{\beta \leq \alpha} \int_{\tilde{U}} A(D^\beta u_j) \cdot M_{\alpha\beta}(y)\, \theta(y)\, dy = \sum_{\beta \leq \alpha} \int_U D^\beta u_j(x)\, M_{\alpha\beta}(\Phi(x)) \cdot \theta(\Phi(x))\, |\det(\Phi'(x))|\, dx$$

$$\| \quad \text{? 纠正} \quad (-1)^{|\alpha|} \int_U \det(\Phi^{-1})(x)\, u_j \cdot D^\alpha(\theta \circ \Phi)(x).\, dx$$

Since $D^\beta u_j \to D^\beta u$ in $L^p(U)$, $\int_{\tilde{U}} |D^\alpha (Au)|^p dy \leq \left(\sum_{\beta \leq \alpha} 1 \right)^p \cdot \max_{|\beta| \leq \alpha} \Big[\sup_{y \in \tilde{U}} |M_{\alpha\beta}| \cdot \int_{\tilde{U}} D^\beta u \cdot |\det$