Extension Theorems

week 3.

Try to preserve weak derivative across the boundary. Warning: $1 \le p \le +\infty$.

*Thm (Extension) Assume U is open, bounded. ∂U is C^1.

for finite p, Lipschitz is suff.

Let V be open bounded s.t. $U \subset\subset V$. Then $\exists$ bounded linear operator

$$E: W^{1,p}(U) \longmapsto W^{1,p}(\mathbb{R}^n)$$

(i) $Eu = u$ a.e. in U

(ii) $\operatorname{supp}(Eu) \subseteq V$.

*(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(U)}$. look like "continous embedding".

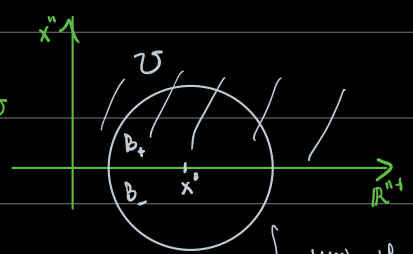
C depends on p, U and V.

Def. We call $E(u)$ an extension of u to $\mathbb{R}^n$. E is called the extension operator.

RMK. if ∂U is C^k, then can construct $Eu \in W^{k,p}(\mathbb{R}^n)$ the proof requires higher order "reflection technique" of the boundary.

Proof.

Simple case: flat ∂U near x^0.



$B^+ := \overline{U} \cap B(x^0, r)$

$B^- := B(x^0, r) \cap \mathbb{R}^n - U$

Assume $u \in C^1(\overline{U})$. Define

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in B_+ \\ -3u(x_1, \cdots, x_{n-1}, -x_n) + 4u(x_1, \cdots, x_{n-1}, -\frac{1}{2}x_n) & \text{if } x \in B_- \end{cases}$$

high order reflection

Idea: if $f(x) = f(0) + f'(0)x + o(x)$ .... ⓪

$f(-x) = f(0) - f'(0)x + o(x)$ .... ①

$f(-\frac{1}{2}x) = f(0) - \frac{f'(0)}{2}x + o(x)$ .... ②

*Claim: $\bar{u} \in C^1(B)$.

Proof. $u^+ := \bar{u}|_{B_+} = u$. $u^- = \bar{u}|_{B_-}$. Then $u^+ = u^-$ on $\partial U = \{x_n = 0\}$.

$-3 \times ① + 4 \times ②$:

$\cdots = f(0) + f'(0)x + o(x)$

$u^+_{x_n} = u_{x_n}$. $u^-_{x_n} = 3u(x_1, \cdots, -x_n) - 2u(x_1, \cdots, \frac{1}{2}x_n)$

has the same 1st order approximation as ⓪.

Can generalize to higher order. Basically linear algebra.

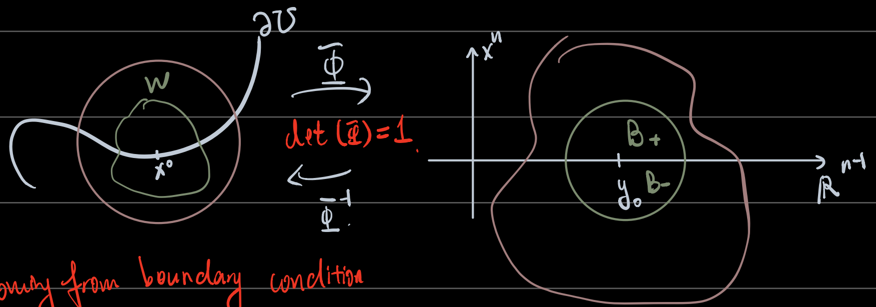
Hence $u^+_{x_n} = u^-_{x_n}$ on ∂U. Also $u^+_{x_i} = u^-_{x_i}$ on ∂U $1 \le i \le n-1$. (*)

The claim follows, since $u \in C^1(\overline{U})$ by assumption.

Hence (*) $\Rightarrow D^\alpha u^+ = D^\alpha u^-$ on $\{x_n = 0\}$. $\forall |\alpha| \le 1$.

Claim: $\|\bar{u}\|_{W^{1,p}(B)} \le C \|u\|_{W^{1,p}(B_+)}$ indep. of u.

Now drop the assumption of flat boundary by "straightening the boundary process".



coming from boundary condition

$\Phi \in C^1$. Let $\tilde{u} := u \circ \Phi^{-1}$ i.e. $\tilde{u}(y) = u(x) = u(\Phi^{-1} y)$

Choose a small ball B centered at y_0 to do extension. Hence $\bar{\tilde{u}} \in C^1(B)$. $\|\bar{\tilde{u}}\|_{W^{1,p}(B)} \le C \|\tilde{u}\|_{W^{1,p}(B_+)}$

Now define $W := \Phi^{-1}(B)$ we get an extension $\bar{u}$ of u into W. $\|\bar{u}\|_{W^{1,p}(W)} \le C \|u\|_{W^{1,p}(U)}$.

$\det \Phi = 1$

Then use compactness of ∂U and partition of unity to get a global extension. Details see Evans.

Note that $Eu := \bar{u}$ is linear. bounded ($\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(U)}$

Finally drop the assumption that $u \in C^1(\bar{U})$ by global approximation theorem: $u_m \in C^\infty(\bar{U})$. $u_m \to u$ in $W^{1,p}(U)$

Then $Eu_m := \overline{u_m}$ $\|Eu_m - Eu_n\|_{W^{1,p}(\mathbb{R}^n)} \le C \|u_m - u_n\|_{W^{1,p}(U)}$ Hence $\{Eu_m\}$ Cauchy in $W^{1,p}(\mathbb{R}^n)$.

Define $Eu := \lim Eu_m$ in the sense of $W^{1,p}(\mathbb{R}^n)$.

Exercise. prove for $W^{k,p}(U)$ $k \ge 2$ provided C^k boundary, using higher order reflection technique.

Extended result (2020. 10. 5)

"Extension of C^∞ functions defined in a half space" R.T. Seely 1963

Given $f \in C^\infty(\mathbb{R}^{n+1}_+)$ with all of its derivatives having continous limits on $\partial \mathbb{R}^{n+1}_+$. Can we extend it to the whole space?

Notation: $\mathbb{R}^{n+1}_+ = \{(x,t) \mid x \in \mathbb{R}^n, t > 0\}$ $D_+ := \{ f \in C^\infty(\mathbb{R}^{n+1}_+) \mid \quad \}$.

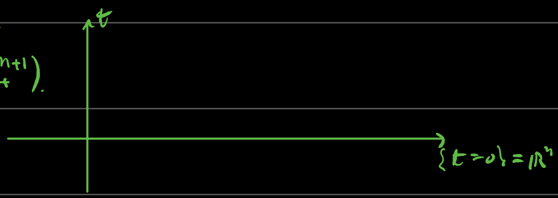
Topology on D_+: unif. conv. of all derivatives.

Theorem (Seely, 1963) There exists a continous linear extension operator $E : D_+ \longrightarrow C^\infty(\mathbb{R}^{n+1})$

s.t. $Ef(x,t) = f(x,t)$ if $t > 0$.

Proof.

$f \in C^\infty(\mathbb{R}^{n+1}_+)$.



Define $Ef(x,t) = \sum_{k=0}^{\infty} a_k \phi(b_k t) f(x, b_k t)$. "Infinite version reflection"

cut off. $\phi \equiv 1$ on $[0,1]$. $\phi \equiv 0$ on $[2,\infty)$ $0 \le \phi \le 1$.

Lemma. $\exists \{a_k\}, \{b_k\}$ s.t. ① $b_k < 0$.

② $\sum |a_k| |b_k|^n < +\infty$ for $n = 0, 1, 2, \cdots$

③ $\sum a_k (b_k)^n = 1$ for $n = 0, 1, 2, \cdots$

④ $b_k \longrightarrow -\infty$ as $k \to \infty$

Q: Why this lemma? A: $t<0$. $Ef(x,t) = \sum_{k=0}^{\infty} a_k \phi(b_k t) f(x, b_k t)$

$\frac{\partial^n Ef}{\partial t^n} = \sum a_k (b_k)^n \cdots$

Proof of Lemma: (same Vandermonder trick) $b_n = -2^n$ Try to solve $\sum_{n=0}^{N} X_n b_n = 1$ for $N = 0, 1, \cdots$

$\Rightarrow a_k^{(N)} = X_k = A_k B_{kN}$. where $A_k = \prod_{j=0}^{k-1} \frac{1+2^j}{2^j - 2^k}$ $B_{kN} = \prod_{j=k+1}^{N} \frac{1+2^j}{2^j - 2^k}$

$\Rightarrow$ $|A_k|$ bounded B_{kN} converge.

$\rightarrow$ $a_k^{(N)}$ converges as $N \to \infty$. ∎

RMK. $W^{k,p}$, $1 \le p \le +\infty$, $k = 0, 1, 2, \cdots$ also works. Extension is cts in the inverse limit topology on

$\tilde{W} := \bigcap_k W^{k,p}$.

Traces (week 4)

Idea: try to assign values to $u \in W^{1,p}(U)$ along $\underbrace{\partial U}_{C^1}$.

Problem: A typical $W^{1,p}$ function is in general not $\in C(\bar{U})$. Even worse, u is defined on U a.e. ∂U has zero n-measure, hence $u|_{\partial U}$ does not make sense.

To solve this bug, use "trace operator". $1 \le p < +\infty$ in this section.

Thm. (Trace theorem) Assume U open, bounded. ∂U is C^1. Then $\exists$ a bounded linear operator $T: W^{1,p}(U) \to L^p(\partial U)$ s.t.

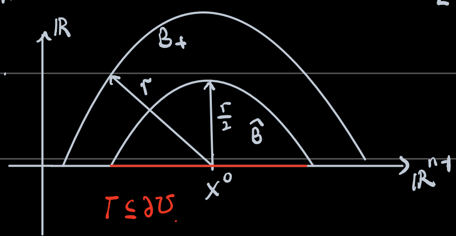
(i) if $u \in W^{1,p} \cap C(\bar{U})$, then $T(u) = u|_{\partial U}$.

(ii) $\|Tu\|_{L^p(\partial U)} \le C_{p,U} \|u\|_{W^{1,p}(U)}$

RMK. We call Tu the trace of u on ∂U.

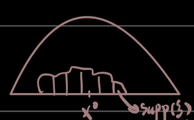
Proof. 1. First assume $u \in C^1(\bar{U})$ by approximation theorem. WANT: $\|Tu\|_{L^p(\partial U)} = \|u\|_{L^p(\partial U)} \lesssim \|u\|_{W^{1,p}(U)}$

For simplicity assume the boundary is flat.



$x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$

$\zeta \in C_c^\infty(B)$. $0 \le \zeta \le 1$. $\zeta \equiv 1$ on $\hat{B}$.



$$\int_{\Gamma=} |u|^p \, dx' \le \int_{\{x_n=0\}} \zeta |u|^p \, dx' \overset{\text{integration by parts}}{=} -\int_{B_+} \partial_{x_n} (\zeta |u|^p) \, dx.$$

Check Evans.

$$= -\int_{B^+} |u|^p \cdot \zeta_{x_n} + \underbrace{\zeta \cdot p \, |u|^{p-1} \mathrm{sgn}(u)}_{\text{Yong's ???}} \cdot u_{x_n} \, dx$$

$$\le C \int_{B_+} |u|^p + |\partial_{x_n} u|^p \, dx \le C \int_{B_+} |u|^p + |Du|^p \, dx \lesssim \|u\|_{W^{1,p}}$$

- If the boundary is not flat, then change of variable $y = \Phi(x)$ to make it flat. $\|u\|_{L^p(\partial U)} \sim \|u \circ \Phi\|_{L^p(\partial \tilde{U})}$, $\|u\|_{W^{1,p}(U)} \sim \|u \circ \Phi\|_{W^{1,p}(\tilde{U})}$
- Now use partition of unity glue things together.

$$\|u\|_{L^p(\Gamma_i)} \le C \|u\|_{W^{1,p}} \quad 1 \le i \le N.$$

Denote $Tu := u|_{\partial U}$. Then $\|Tu\|_{L^p(\partial U)} \le \sum_{i=1}^{N} \|u\|_{L^p(\Gamma_i)} \le C \cdot \|u\|_{W^{1,p}(U)}$.

2. Approximation. Assume $u \in W^{1,p}$. Take $\{u_m\} \subseteq C^\infty(\bar{U})$, $u_m \longrightarrow u$ in $W^{1,p}$.

Then $\|Tu_m\|_{L^p(\partial U)} \le C\|u_m\|_{W^{1,p}}$. Hence $\{Tu_m\}_{m=1}^\infty$ is Cauchy in $L^p(\partial U)$.

By completeness, take $Tu := \lim Tu_m$ in the L^p sense.

Then $\|Tu\|_{L^p(\partial U)} = \lim \|Tu_m\| \le C \lim \|u_m\|_{W^{1,p}(U)} = C\|u\|_{W^{1,p}(U)}$ ∎

RMK. if $u \in W^{1,p}(U) \cap C(\bar{U})$, then $u_m \in C^\infty(\bar{U}) \Longrightarrow u$ on $\bar{U}$

then $Tu := u|_{\partial U}$

Thm. Characterization of Trace-zero functions in $W^{1,p}(U)$.

Assume U bounded with C^1 ∂U. Suppose $u \in W^{1,p}(U)$. Then.

$u \in \underbrace{W_0^{1,p}(U)}_{\text{completion of } C_c^\infty \text{ in } W^{1,p}} \Longleftrightarrow Tu = 0$ on ∂U.

Proof. 1. $\Rightarrow$ Suppose $u \in W_0^{1,p}(U) \overset{def}{\Longleftrightarrow} \exists\, u_m \in C_c^\infty(U) \to u$ in $W^{1,p}$

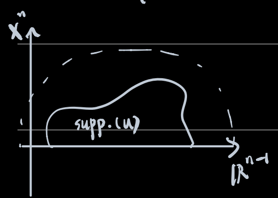
Then $Tu_m = u_m|_{\partial U} = 0$. also T is bounded linear operator.

Hence $\|Tu\|_{L^p(\partial U)} = \lim \|Tu_m\|_{L^p(\partial U)} = 0$.

2. $\Leftarrow$: The decay estimate along x_n-direction.

Assume ∂U is flat as usual. Do local argument and then use partition of unity as usual.

Assume $\Longrightarrow \begin{cases} u \in W^{1,p}(\mathbb{R}^n_+ := \{x_n > 0\}) \text{ has compact support in } \overline{\mathbb{R}^n_+}. \\ Tu = 0 \text{ on } \{x_n = 0\} = \mathbb{R}^{n-1} \end{cases}$



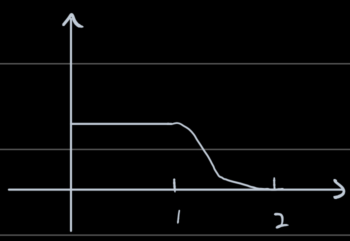
$Tu = 0$ on $\{x_n = 0\} \Longrightarrow \exists\, u_m \in W^{1,p}(\mathbb{R}^n_+) \cap C^1(\overline{\mathbb{R}^n_+})$ s.t. $\begin{cases} u_m \to u \text{ in } W^{1,p}(\mathbb{R}^n_+) \\ Tu_m = u_m|_{x_n=0} \longrightarrow 0 \text{ in } L^p(\mathbb{R}^{n-1}) \end{cases}$

Now if $x' = (x_1, \cdots, x_{n-1})$ $\|u_m(x', x_n)\| \le |u_m(x', 0)| + \left|\int_0^{x_n} |\partial_{x_n} u_m(x', t)|\, dt\right|$

$$\Longrightarrow \int_{\mathbb{R}^{n-1}} |u_m(x',x_n)|^p dx' \le C\cdot\Big[\underbrace{\int_{\mathbb{R}^{n-1}} |u_m(x',0)|^p dx'}_{\longrightarrow 0 \text{ since } Tu_m\to 0.} + x_n^{p-1}\underbrace{\int_0^{x_n}\int_{\mathbb{R}^{n-1}} |D u_m(x',t)|^p dx'\,dt.}_{\text{looks like } \|Du_m\|^p_{L^p(\mathbb{R}^n_+)}}\Big]$$

Send $m\to\infty$. $\underbrace{\int_{\mathbb{R}^{n-1}} |u(x',x_n)|^p dx'}_{\|Tu\|^p_{L^p(\partial\mathbb{R}^n_+)}} \le C\, x_n^{p-1}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} |Du|^p dx'\,dt.$ $(*)$ holds for a.e. $x_n>0$.

- Let $\xi\in C^\infty(\mathbb{R}_+)$ be a cut off function: $\xi\equiv 1$ on $[0,1]$. $\xi\equiv 0$ on $[2,+\infty)$. $0\le\xi\le 1$.



$\xi_m(x) = \xi(m x_n)$ $x\in\mathbb{R}^+_n$

$w_m = u(x)\,(1-\xi_m(x))$

Then $w_m\equiv u$ on $\{x_n \ge 2\frac{1}{m}\}$

$w_m\equiv 0.$ on $\{x_n\le\frac{1}{m}\}$

$\partial_{x_n} w_m = \partial_{x_n} u\,(1-\xi_m) - m\,u\,\xi'(m x_n)$

$D_{x'} w_m = Du\cdot(1-\xi_m)$.

$$\int_{\mathbb{R}^n_+} |Dw_m - Du|^p dx \le C\underbrace{\int_{\mathbb{R}^n_+} |\xi_m|^p\,|Du|^p dx}_{A} + C\,m^p\underbrace{\int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}} |u|^p dx'\,dt}_{B}$$

A is fine. $\mathrm{supp}(\xi_m)\subseteq\{0\le x_n\le\frac{2}{m}\}\longrightarrow 0.$ Hence $A\to 0$.

B: By using decay estimate $(*)$: $B\le m^p.\left(\int_0^{\frac{2}{m}} t^{p-1}dt\right)\left(\int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}}|Du|^p dx'\,dt\right)$

$\le C\int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}}|Du|^p dx'\,dt \longrightarrow 0$ since $\frac{2}{m}\to 0$.

$\Longrightarrow w_m\to u$ in $W^{1,p}(\mathbb{R}^+_n)$.

$w_m = 0$ if $0\le x_n<\frac{1}{m}$

Then mollify w_m to produce $u_m\in C_c^\infty(\mathbb{R}^+_n)$ s.t. $u_m\to u$ in $W^{1,p}(\mathbb{R}^+_n)$.

$\overset{\text{def.}}{\Longrightarrow} u\in W_0^{1,p}(\mathbb{R}^n_+)$

* Sobolev Embedding Inequalities week 5

Theorem (Gagliardo–Nirenburg–Sobolev)

Assume $f \in C_c^\infty(\mathbb{R}^n)$ then $\|f\|_{L^q(\mathbb{R}^n)} \lesssim_{n,m,p} \|\partial^m f\|_{L^p(\mathbb{R}^n)}$

where $\frac{1}{q} = \frac{1}{p} - \frac{m}{n} > 0$. $1 \leq p < +\infty$

When $q = \infty$: $\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim_{n,m,p} \sum_{k=0}^{m} \|\partial^k f\|_{L^p(\mathbb{R}^n)}$ where $m > \frac{n}{p}$

RMK. (Non sharp version) If $1 \leq p < q < \infty$. $\frac{1}{p} - \frac{m}{n} < \frac{1}{q}$. then $\|f\|_{L^q(\mathbb{R}^n)} \leq C_{m,n,p,q} \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}$.

Consider a special case $1 \leq p < n$: $u \in C_c^\infty(\mathbb{R}^n)$ $\|u\|_{L^q} \lesssim \|Du\|_{L^p(\mathbb{R}^n)}$. $\underbrace{\frac{1}{q} = \frac{1}{p} - \frac{1}{n}}_{\text{so called "sobolev conjugate"}}$. $(p < q < n)$

By dimension analysis. $u_\lambda(x) = u(\lambda x)$ $(\lambda > 0)$. Assume the ineq. works for u, it works for u_λ as well:

$$\left.\begin{aligned} &\int_{\mathbb{R}^n} |u_\lambda|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \lambda^{-n} \int |u(y)|^q dy \\ &\int |Du_\lambda|^p dx = \lambda^p \int |Du(\lambda x)|^p dx = \lambda^{\frac{p}{n}} \int |Du|^p dx \end{aligned}\right\} \Rightarrow \lambda^{-\frac{n}{q}} \|u\|_{L^q} \lesssim \lambda \cdot \lambda^{-\frac{n}{p}} \|Du\|_{L^p}$$

So it must work for all $\lambda > 0$. The index of λ must vanish: $-\frac{n}{q} = 1 - \frac{n}{p} \Leftrightarrow \frac{1}{p} - \frac{1}{q} = \frac{1}{n}$.

Proof of the ineq. for $m=1$. $\|u\|_q \lesssim \|Du\|_p$ $q = \frac{np}{n-p}$. $1 \leq p < n$. $u \in C_c^1(\mathbb{R}^n)$

1) We first prove it for $p = 1$: $\|u\|_{\frac{n}{n-1}} \lesssim \|Du\|_{L^1}$

① Dimension 1: $\|u\|_{L^\infty(\mathbb{R})} \lesssim \|u'(x)\|_{L^1}$. $u \in C_c^1(\mathbb{R})$.

Write $u(x) = \int_{-\infty}^x u'(y)\,dy$ Done.

② Dimension 2: $q = 2$. Do it for each variable, and organize it nicely.

$$\left.\begin{aligned} &|f(x_1, x_2)| \lesssim \int |\partial_1 f(y_1, x_2)|\, dy_1 \\ &|f(x_1, x_2)| \lesssim \int |\partial_2 f(x_1, y_2)|\, dy_2 \end{aligned}\right\} \text{by ①}.$$

$\Rightarrow |f|^2 \lesssim \int |\partial_1 f(y_1, x_2)|\, dy_1 \int |\partial_2 f(x_1, y_2)|\, dy_2$

$\Rightarrow \int_{\mathbb{R}^2} |f(x_1, x_2)|\, dx_1 dx_2 \lesssim \|\partial_1 f\|_{L^1(\mathbb{R}^2)} \|\partial_2 f\|_{L^1(\mathbb{R}^2)}$

$\Rightarrow \|f\|_{L^2(\mathbb{R}^2)} \lesssim \underbrace{\|Df\|_{L^1(\mathbb{R}^2)}}_{= \| \sqrt{|\partial_1 f|^2 + |\partial_2 f|^2} \|_{L^1}}$ Done.

③ Dimension $n \geq 3$: tricky. Assume $n=3$ (most representative case) $q = \frac{3}{2}$.

$$|f(x)|^{\frac{3}{2}} \lesssim \underbrace{\left(\int_{\mathbb{R}} |\partial_1 f(y_1, x_2, x_3)| dy_1\right)^{\frac{1}{2}}}_{\int_1 f_1} \underbrace{\left(\int |\partial_2 f(x_1, y_2, x_3)| dy_2\right)^{\frac{1}{2}}}_{\int_2 f_2} \underbrace{\left(\int |\partial_3 f(x_1, x_2, y_3)| dy_3\right)^{\frac{1}{2}}}_{\int_2 f_3}$$

Denote $\int_1 f_1$ $\quad \int_2 f_2 \quad \int_2 f_3$.

$$\Longleftrightarrow \quad |f(x)|^{\frac{3}{2}} \lesssim \left(\int_1 |f_1(\cdot, x_2, x_3)|\right)^{\frac{1}{2}} \left(\int_2 |f_2(x_1, \cdot, x_3)|\right)^{\frac{1}{2}} \left(\int_3 |f_3(x_1, x_2, \cdot)|\right)^{\frac{1}{2}}$$

$$\int_1 |f(\cdot, x_2, x_3)|^{3/2} dx_1 \lesssim \left(\int_1 |f_1(\cdot, x_2, x_3)|\right)^{\frac{1}{2}} \int_1 \left(\int_2 |f_2|\right)^{\frac{1}{2}} \left(\int_3 |f_3|\right)^{\frac{1}{2}} dx_1$$

$$\overset{\text{Hölder}}{\lesssim} \left(\int_1 |f_1(\cdot, x_2, x_3)|\right)^{\frac{1}{2}} \left(\int_{1,2} |f_2(\cdot, \cdot, x_3)|\right)^{\frac{1}{2}} \left(\int_{1,3} |f_3(\cdot, x_2, \cdot)|\right)^{\frac{1}{2}}$$

$$\Rightarrow \int_{1,2} |f|^{3/2} \overset{\text{Hölder}}{\lesssim} \left(\int_{1,2} |f_1|\right)^{?} \left(\int_{1,2} |f_2|\right)^{?} \left(\int_{1,3,2} |f_3|\right)^{?}$$

$$\int_{1,2,3} |f|^{3/2} \overset{\text{Hölder}}{\lesssim} \left(\int_{1,2,3} |f_1|\right)^{?} \left(\int_{1,2,3} |f_2|\right)^{?} \left(\int_{1,2,3} |f_3|\right)^{?}$$

In general. $\|f\|_{\frac{n}{n-1}} \lesssim \prod_{i=1}^{n} \|\partial_i f\|_{L^1}^{\frac{1}{n}}$

2) For $1 < p < n$: deduce from $\|f\|_{\frac{n}{n-1}} \lesssim \|Df\|_{L^1}$ ($p=1$ case)

Apply the above ineq. to $f = |u|^{\gamma} \in C_c^1$:

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}} \lesssim \int_{\mathbb{R}^n} D(|u|^{\gamma})\, dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du|\, dx$$

$$\overset{\text{Hölder}}{\lesssim} \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\right)^{\frac{1}{p}}$$

Choose γ s.t. $\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} \Longleftrightarrow \gamma = \frac{p(n-1)}{n-p} > 1$.

$$\frac{\gamma n}{n-1} = \frac{np}{n-p} = q$$

$$\Rightarrow \|u\|_q \lesssim \|Du\|_p$$

$C = \frac{p(n-1)}{n-p}$ const. not sharp. blow up when $p = n$.

Now for higher order derivative $m > 1$. $\quad \|f\|_{L^q(\mathbb{R}^n)} \lesssim_{m,n,p} \|\partial^m f\|_{L^p(\mathbb{R}^n)} \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{n} > 0.$

we apply the $m=1$ case.

$$\|f\|_{L^q} \lesssim \|\partial f\|_{\left(\frac{1}{q}+\frac{1}{n}\right)^{-1}} \lesssim \|\partial^2 f\|_{\left(\frac{1}{q}-\frac{2}{n}\right)^{-1}} \lesssim \cdots \lesssim \|\partial^m f\|_{L^p}.$$

Theorem (Estimates for $W^{1,p}$, $1 \le p < n$). Bounded domain.

Assume U bounded open. ∂U is C^1, $u \in W^{1,p}(U)$. then: $L^q(U)$ where $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$.

$$\|u\|_{L^q} \lesssim_{n,p,U} \|u\|_{W^{1,p}}$$

Proof: By extension thm. $\exists\, \bar{u} \in W^{1,p}(\mathbb{R}^n)$ $\bar{u}$ has compact supp. $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,p}(U)}$

$\exists\, u_m \in C_c^\infty(\mathbb{R}^n)$ $u_m \to \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$

$$\|u_m - u_\ell\|_{L^q(U)} \lesssim \|Du_m - Du_\ell\|_{L^p}$$

$\Rightarrow$ LHS conv. in L^q.

$$\|\bar{u}\|_{L^q} \lesssim \|D\bar{u}\|_{L^p}$$

$\Rightarrow \|u\|_{L^q} \lesssim \|Du\|_{L^p}$ and hence done.

Theorem (Estimates for $W_0^{1,p}$ $1 \le p < n$)

Assume U bounded open in $\mathbb{R}^n$. $u \in W_0^{1,p}$ $1 \le p < n$. Then $\|u\|_{L^q} \lesssim_{p,q,n,U} \|Du\|_{L^p}$ where $q \in [1, p^*]$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

In particular, $\|u\|_{L^p} \lesssim \|Du\|_{L^p}$ $\forall\, 1 \le p < n$. (Poincaré Inequality)

$\|u\|_{W^{1,p}} \sim \|Du\|_{L^p} \sim \|u\|_{L^p}$

Proof. $u \in W_0^{1,p} \Rightarrow \exists\, u_m \in C_c^\infty(U)$ $u_m \to u$ in $W^{1,p}(U)$. Define $u_m \equiv 0$ on $\mathbb{R}^n - \bar{U}$

By previous thm. we can extend u to $\mathbb{R}^n$.

$\Rightarrow \|u\|_{L^{p^*}} \lesssim \|Du\|_{L^p}$

Now since U is bounded, $|U| < \infty$. $\xrightarrow{\text{Hölder}} \|u\|_{L^q} \lesssim \|u\|_{L^{p^*}}$ $\forall\, q \in [1, p^*]$.

RMK. The borderline case: $p = n$. $p^* = \frac{np}{n-p} = +\infty$. $\|u\|_{L^\infty} \lesssim \|u\|_{W^{1,n}}$ is true only for $n = 1$.

$n > 1$: $u \in W^{1,n} \not\Rightarrow u \in L^\infty$.

Counter example: $U = B(0,1) = \{x \in \mathbb{R}^n \mid |x| < 1\}$ $u = \log(\log(1 + \frac{1}{|x|}))$

$u \in W^{1,n}(U)$ (using polar coordinate) But $u \notin L^\infty$. ($u \in$ BMO only)

week 6

Thm (Hardy-Littlewood-Sobolev ineq.) Let $0<\gamma<n$. $1<p<q<\infty$. $\underbrace{1-\frac{\gamma}{n}=\frac{1}{p}-\frac{1}{q}}_{\text{scaling relation}}$.

Then $\| |\cdot|^{-\gamma} * f(x) \|_{L^q(\mathbb{R}^n)} \lesssim_{p,q,n,\gamma} \|f\|_{L^p(\mathbb{R}^n)}$

Proof. $(I_\gamma f)(x) := (|\cdot|^{-\gamma} * f)(x)$ WLOG $f\geq 0$.

$$= \int_{|y|>R} \frac{f(x-y)}{|y|^\gamma} dy + \int_{|y|\leq R} \frac{f(x-y)}{|y|^\gamma} dy$$

$$\overset{\text{Hölder}}{\leq} \|f\|_{L^p} \left(\int_{|y|\geq R} |y|^{-\gamma p'} dy \right)^{\frac{1}{p'}}$$

$$\lesssim R^{\frac{n}{p'}-\gamma} \|f\|_{L^p}$$ Need: $\gamma p' > n$. holds if $q<+\infty$. since $1-\frac{\gamma}{n}=\frac{1}{p}-\frac{1}{q} \Leftrightarrow \frac{\gamma}{n} = \frac{1}{p'}+\frac{1}{q}$

Use maximal function: $\int_{|y|<R} \frac{f(x-y)}{|y|^\gamma} dy \overset{\text{Dyadic decomp.}}{\leq} \sum_{k=0}^{\infty} \int_{2^{-k-1}\leq|y|\leq 2^{-k}R} \frac{f(x-y)}{|y|^\gamma} dy$

$$\lesssim \sum_{k=0}^{\infty} \frac{1}{(2^{-k}R)^\gamma} \int_{|y|\leq 2^{-k}R} |f(x-y)| dy$$

$$\lesssim \sum_{k=0}^{\infty} (2^{-k}R)^{n-\gamma} \cdot (Mf)(x)$$ Need: $\gamma<n$ to make Σ converges

$$\lesssim R^{n-\gamma} (Mf)(x)$$

Now optimize R: $R^{n-\gamma} Mf(x) = R^{\frac{n}{p'}-\gamma} \|f\|_{L^p}$.

$$\Rightarrow R(x) = \left(\frac{\|f\|_{L^p}}{(Mf)(x)} \right)^{\frac{p}{n}}$$

$$\Rightarrow (I_\gamma f)(x) \lesssim \|f\|_{L^p}^{\frac{p}{n}(n-\gamma)} |Mf(x)|^{1-\frac{p}{n}(n-\gamma)}$$

The scaling relation) $1-\frac{\gamma}{n}=\frac{1}{p}-\frac{1}{q} \Leftrightarrow \frac{p}{n}(n-\gamma) = 1-\frac{p}{q}$

$$\Rightarrow \quad \lesssim \|f\|_{L^p}^{1-\frac{p}{q}} |Mf(x)|^{\frac{p}{q}}$$

$$\Rightarrow |(I_\gamma f)(x)|^q \lesssim \|f\|_{L^p}^{q-p} |Mf(x)|^p.$$

$$\|I_\gamma f\|_{L^q} \lesssim \|f\|_{L^p}^{1-\frac{p}{q}} \|Mf\|_{L^p}^{\frac{p}{q}} \underset{\text{Need } 1<p.}{\lesssim} \|f\|_{L^p}$$

RMK. H-L-S ineq. has a simple bilinear formulation.

$$\iint \frac{f(x)g(y)}{|x-y|^\gamma} dxdy \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$ where $0<\gamma<n$. $0<p_1,p_2<\infty$. $\frac{1}{p_1'}+\frac{1}{p_2'}=\frac{\gamma}{n}$ or $2=\frac{1}{p}+\frac{1}{q}+\frac{\gamma}{n}$

Alternative proof for G-N-S ineq. with $1<p<n$. $\|u\|_{\frac{np}{n-p}} \lesssim \|Du\|_{L^p}$ using H-L-S. ineq.

Assume $u\in C_c^\infty(\mathbb{R}^n)$. take unit vector $\omega\in\mathbb{R}^n$.

$$u(x) = -\int_0^\infty \frac{d}{dr} u(x+\omega r) dr$$

$$|u(x)| \le \int_{|\omega|=1}\int_0^\infty \left|\frac{d}{dr}u(x+\omega r)\right| \underline{dr\, d\sigma(\omega)}$$

$dy = r^{n-1}\, dr\, d\sigma(\omega)$. Coarea - formula ?

$$\lesssim \int_{\mathbb{R}^n} \frac{|Du|(y)}{|x-y|^{n-1}}\, dy$$

$$= \left(|\cdot|^{-(n-1)} * Du \right)(x)$$

HLS Ineq. $\Rightarrow \| \, |\cdot|^{1-n} * Du \|_{L^q} \lesssim \|Du\|_{L^p}$. $\quad p > 1. \quad 1 - \frac{n-1}{n} = \frac{1}{p} - \frac{1}{q} \Rightarrow q = \frac{np}{n-p}$. ■

Coarea Formula $u: \mathbb{R}^n \to \mathbb{R}$ Lipschitz. Assume for a.e. $r \in \mathbb{R}$, $\{x \in \mathbb{R}^n \mid u(x) = r\} = u^{-1}(r)$ is a smooth $n-1$ dimension hypersurface.

Suppose $f: \mathbb{R}^n \to \mathbb{R} \in L^1(\mathbb{R}^n)$, cts. Then $\displaystyle\int_{\mathbb{R}^n} f \cdot |Du|\, dx = \int_{-\infty}^{\infty} \left(\int_{\{u=r\}} f\, dS \right) dr$

Corollary. In polar coordinates,

(i) Let $f: \mathbb{R}^n \to \mathbb{R}$ be cts and L^1. then $\displaystyle\int_{\mathbb{R}^n} f\, dx = \int_0^\infty \left(\int_{\partial B(x_0, r)} dS \right) dr$. $\forall x_0 \in \mathbb{R}^n$.

(ii) $\displaystyle\frac{d}{dr}\left[\int_{B(x_0,r)} f\, dx \right] = \int_{\partial B(x_0, r)} f\, dS$ for each $r > 0$.

RMK. Hilbert Inequality.

Trace thm for half space

Thm. Let $s > \frac{1}{2}$. Then $\exists$ a cts linear map $T: H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$, which we call the trace operator s.t. $\forall$ smooth f, $T(f)(x_1, \cdots, x_{n-1}) = f(x_1, \cdots, x_{n-1}, 0)$

Proof by Fourier transf. take f smooth. $g(x') = f(x', 0)$

$\tilde{f}$ = Fourier transf. of f in x_n variable only

$\hat{f}$ = Fourier transf. of f in $\mathbb{R}^n$

$\hat{g}$ = ... g in $\mathbb{R}^{n-1}$

$$\tilde{f}(x', \xi_n) = \int_{-\infty}^{\infty} f(x', x_n) \exp(-i x_n \xi_n)\, dx_n.$$

$\Rightarrow$ Use Fourier inversion at $x_n = 0$:

$$g(x') = f(x', 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(x', \xi_n)\, d\xi_n$$

$$\hat{g}(\xi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi', \xi_n)\, d\xi_n$$

$$\|g\|_{H^{s-\frac{1}{2}}} \lesssim \int_{\mathbb{R}^{n-1}} |\hat{g}(\xi')|^2 (1+|\xi'|^2)^{s-\frac{1}{2}}\, d\xi'$$

$$\lesssim \int_{\mathbb{R}^{n-1}} \left| \int_{-\infty}^{+\infty} \hat{f}(\xi', \xi_n) d\xi_n \right|^2 (1+|\xi'|^2)^{s-\frac{1}{2}}\, d\xi'.$$

Cauchy-Schwartz

$$\lesssim \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{+\infty} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s\, d\xi_n \right) \left(\int_{-\infty}^{+\infty} (1+|\xi|^2)^{-s}\, d\xi_n \right) (1+|\xi'|^2)^{s-\frac{1}{2}}\, d\xi'$$

$$\int_{-\infty}^{+\infty} (1+|\xi|^2)^{-s}\, d\xi_n = \int_{-\infty}^{+\infty} (1+|\xi'|^2+\xi_n^2)^{-s}\, d\xi_n = (1+|\xi'|)^{-s+\frac{1}{2}} \int_{-\infty}^{+\infty} (1+y^2)^{-s}\, dy$$

$$\Rightarrow \|g\|^2_{H^{s-\frac{1}{2}}} \lesssim \|f\|^2_{H^s} \qquad \blacksquare$$