Morrey Embedding

Thm. $W_0^{1,p}(U) \hookrightarrow \begin{cases} L^{\frac{np}{n-p}}(U) & \text{if } p<n \\ C^0(\overline{U}) & \text{if } p>n \end{cases}$

open bounded.

Furthermore, (1) $\|u\|_{\frac{np}{n-p}} \lesssim \|Du\|_p$, $p<n$ already proved $\forall u \in W_0^{1,p}(U)$

(2) $\underbrace{\sup_{x\in U} |u|}_{?\ \|u\|_\infty} \lesssim |U|^{\frac{1}{n}-\frac{1}{p}} \|Du\|_p$, $p>n$.

Proof of (2): Assume $u \in C_0^1(U)$. Recall $|u|^{\frac{n}{n-1}} \lesssim \left(\prod_{i=1}^n \int_{-\infty}^{+\infty} |\partial_i u|\, dx_i\right)^{\frac{1}{n-1}}$

$$\Rightarrow \|u\|_{\frac{n}{n-1}} \lesssim \left(\prod_{i=1}^n \int_U |\partial_i u|\, dx\right)^{\frac{1}{n}} \overset{\text{AM-GM}}{\leq} \frac{1}{n}\int_U \sum_{i=1}^n |\partial_i u|\, dx$$

$$\sum|\partial_i u| \leq \sqrt{n}\left(\sum|\partial_i u|^2\right)^{\frac{1}{2}} \qquad \leq \frac{1}{\sqrt{n}} \|Du\|_{L^1(U)}$$

• $\gamma>1$. $\| |u|^\gamma \|_{\frac{n}{n-1}} \leq \frac{\gamma}{\sqrt{n}} \int_U |u|^{\gamma-1} |Du|\, dx$

$$\overset{\text{Hölder}}{\leq} \frac{\gamma}{\sqrt{n}} \| |u|^{\gamma-1} \|_{p'} \|Du\|_p$$ Our assumption. $Du \in L^p$. $p>n$.

Define $\tilde{u} := \dfrac{\sqrt{n}\,|u|}{\|Du\|_p}$ Then $\boxed{\| |\tilde{u}|^\gamma \|_{\frac{n}{n-1}} \leq \gamma \| |\tilde{u}|^{\gamma-1} \|_{p'}}$ $(*)$ $\forall \gamma>1$.

For simplicity, assume $|U| = 1$.

$$\|\tilde{u}\|_{\gamma\frac{n}{n-1}} \leq \gamma^{\frac{1}{\gamma}} \|\tilde{u}\|_{p'(\gamma-1)}^{1-\frac{1}{\gamma}} \qquad \forall \gamma>1.$$

$$\leq \gamma^{\frac{1}{\gamma}} \|\tilde{u}\|_{p'\gamma}^{1-\frac{1}{\gamma}} \quad \text{since } |U|=1.$$

$p>n \Rightarrow n'>p'$ Higher norm controlled by lower norm.

$A = \frac{n'}{p'} > 1$. $\gamma_n = A^n$.

$$\Rightarrow \boxed{\|\tilde{u}\|_{A^{m+1}\cdot\frac{n}{n-1}} \leq (A^m)^{\frac{1}{A^m}} \|\tilde{u}\|_{A\cdot p'}^{1-\frac{1}{A^m}}}$$ Iteration procedure

$$\Rightarrow \|\tilde{u}\|_{A^{m+1}\cdot\frac{n}{n-1}} \leq A^{\sum_{j=0}^{m}\frac{j}{A^j}}$$ Note $A>1$.

$$\left(\text{Initial Step: } \|\tilde{u}\|_{\frac{n}{n-1}} \leq 1.\right)$$

$$\Rightarrow \|\tilde{u}\|_{\infty} \leq \underbrace{A^{\sum_{j=0}^{\infty}\frac{j}{A^j}}}_{\text{Call it } C_1(n,p)} < +\infty$$

$$\Rightarrow \|u\|_{\infty} \leq \frac{C_1(n,p)}{\sqrt{n}}\|Du\|_p \qquad (|U|=1,\ p>n).$$

Now for general U with $|U|>0$. make a change of variable.

$$y_i = |U|^{\frac{1}{n}}x_i \Rightarrow |\tilde{U}|=1.$$

$$\Rightarrow \|u\|_{\infty} \lesssim \frac{C_1(n,p)}{\sqrt{n}}|U|^{\frac{1}{n}-\frac{1}{p}}\|Du\|_p.$$

RMK. For $p<n$. the const. we computed is not optimal. The sharp const. is $C=\frac{1}{n\sqrt{\pi}}\left(\frac{n!\,\Gamma(\frac{n}{2})}{2\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}\right)^{\frac{1}{n}}\cdot\left(\frac{n(p-1)}{n-p}\right)^{1-\frac{1}{p}}$

Def. A Banach Space B_1 is said to be continuously embedded into a Banach space B_2. (write as $B_1 \hookrightarrow B_2$.) if $\exists$ a bounded linear 1-1 map $T: B_1 \to B_2$.

Corollary. $k\geq 0$ is an integer. $W_0^{k,p}(U) \hookrightarrow \begin{cases} L^{\frac{np}{n-kp}} & \text{if } kp<n \\ C^m(\overline{U}) & \text{if } 0\leq m<kp. \end{cases}$

Proof. Iterate k times!

RMK. Working on $W_0^{k,p}$, we don't need regularity of ∂U

RMK. $\|u\|_{W_0^{k,p}} \sim \left(\int_U \sum_{|\alpha|=k}|D^\alpha u|^p dx\right)^{\frac{1}{p}}$

Thm. (Replace $W_0^{k,p}$ by $W^{k,p}$) Suppose U satisfys "uniform interior cone condition". Then

$$W^{k,p}(U) \hookrightarrow \begin{cases} L^{\frac{np}{n-kp}}(U) & \text{if } kp<n \\ C_B^m(U) & \text{if } 0\le m<k-\frac{n}{p} \end{cases}, \text{ where } C_B^m(U) := \{u\in C^m(U) \mid D^\alpha u \in C^\infty(U),\ \forall |\alpha|\le m\}.$$

Def. uniform interior cone condition: $\exists$ a fixed cone K_U s.t. $\forall x\in U$ is a vertex of a cone $K_U(x) \subseteq \overline{U}$, and $K_U \overset{全等}{\cong} K_U(x)$.

General conditions for the domain:

key requirement. ∂U must be $n-1$ dimensional, and U lies only one side of its boundary.

- Uniform C^m-regularity condition. $(m\ge 2)$.

$\Rightarrow$ strong local lipschitz condition

$\Rightarrow$ uniform cone condition.

$\Rightarrow$ the segment condition.

- uniform cone condition $\Rightarrow$ cone condition

RMK. Typical Sobolev Embedding Thm (Except C^j, $C^{j,\lambda}$). can be proved under cone condition.

week 7

Lemma Representation formula for $W_0^{1,1}(\Omega)$

$u \in W_0^{1,1}(\Omega)$ Then $u(x) = \dfrac{1}{n\,\omega_n} \displaystyle\int_\Omega \frac{(x-y)\cdot \nabla u}{|x-y|^n}\, dy$, a.e. $x \in \Omega$.

$\underbrace{n\,\omega_n}_{\text{area measure of } S^{n-1}}$, ω_n: volume of unit ball

RMK higher order formula if $u \in C^2(\Omega)$:

$$u(x) = \int_\Omega \Phi(x-y)\cdot(-\Delta u)(y)\, dy$$

where $\Phi(x) = \begin{cases} \frac{1}{n(n-1)} |x|^{-(n-2)} & n > 2 \\ \frac{1}{2\pi}(-1) \lg|x| & n = 2. \end{cases}$

Def. $\mu \in (0,1]$. define the Riesz potential V_μ on $L^1(\Omega)$ $(V_\mu f)(x) = \displaystyle\int_\Omega |x-y|^{n(\mu-1)} f(y)\, dy$

RMK. If $f \equiv 1$, Fact. $V_\mu 1 \le \mu^{-1} \omega_n^{1-\mu} |\Omega|^\mu$.

Proof. Chose $R > 0$ s.t. $|\Omega| = \mu(B_R(x)) = \omega_n R^n$

$$\int_\Omega |x-y|^{n(\mu-1)} dy \le \int_{B_R(x)} |x-y|^{n(\mu-1)} dy = R^{n\mu} \cdot \omega_n \cdot \mu^{-1} = \mu^{-1} \omega^{1-\mu} |\Omega|^\mu \quad \blacksquare$$

Lemma. V_μ maps $L^p(\Omega)$ into $L^q(\Omega)$ continuously (bounded). for $1 \le q \le +\infty$. $0 \le \delta := \frac{1}{p} - \frac{1}{q} < \mu$

$$\|V_\mu f\|_q \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_p.$$

Proof. Choose $r \ge 1$ s.t. $r^{-1} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \delta$

Consider $h(x-y) = |x-y|^{n(\mu-1)} \in L^r(\Omega)$

$$\Rightarrow \|h\|_{L^r} \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta}$$

Then we mimic the usual proof of Young's Inequality for convolution.

$h|f| = h^{\frac{r}{q}} h^{r(1-\frac{1}{p})} |f|^{\frac{p}{q}} |f|^{p\delta}$ Then use Hölder

$$|V_\mu f| \le \left(\int_\Omega h^r(x-y)|f(y)|^p dy\right)^{\frac{1}{q}} \cdot \left(\int_\Omega h^r(x-y)\right)^{1-\frac{1}{p}} \left(\int_\Omega |f(y)|^p dy\right)^\delta$$

$$\|V_\mu f\|_q \le \sup\left\{\int_\Omega h^r(x-y)dy\right\}^{\frac{1}{r}} \|f\|_p$$

$$\le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_p \quad \blacksquare$$

Lemma. Let $f \in L^p(\Omega)$, $g = V_{\frac{1}{p}} f$. Then

$$\int_\Omega \exp\left(\frac{g}{C_1 \|f\|_p}\right)^{p'} dx \le C_2 |\Omega|.$$

where $p' = \frac{p}{p-1}$, C_1, C_2 depend on (p, n).

Proof. By previous lemma, $\|g\|_q \le q^{1-\frac{1}{p}+\frac{1}{q}} \omega_n^{1-\frac{1}{p}} |\Omega|^{\frac{1}{q}} \|f\|_p \quad \forall q \ge p.$

$$\Rightarrow \int_\Omega |g|^q dx \le q^{1+\frac{q}{p'}} \omega^{\frac{q}{p'}} |\Omega| \, \|f\|_p^q \quad \forall q \ge p$$

$q \to p'q$

$$\Rightarrow \int_\Omega |g|^{\underbrace{p'q}_{\ge p}} \le p'q \left(\omega_n p'q \|f\|_p^{p'}\right)^q |\Omega|$$

$$\int_\Omega \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{|g|}{C_1 \|f\|_p}\right)^{p'k} dx \le p' |\Omega| \sum \left(\frac{p'\omega_n}{C_1 p'}\right)^k \frac{k^k}{(k-1)!}$$ Stirling formula.

Then apply apply Lebesgue monotone convergence Thm.

Take C_1 large enough

Theorem. Suppose $u \in W_0^{1,n}(\Omega)$. Then $\exists\, C_1(n), C_2(n)$ s.t.

$$\int_\Omega \exp\left(\frac{|u|}{C_1 \|Du\|_n}\right)^{\frac{n}{n-1}} dx \le C_2 |\Omega|.$$

Lemma. (Morrey) Let Ω be convex. Assume $u \in W^{1,1}(\Omega)$.

Then $$|u(x) - u_S| \le \frac{d^n}{n|S|} \int_\Omega |x-y|^{1-n} |Du(y)| \, dy \quad \text{a.e. } x \in \Omega$$

$u_S = \frac{1}{|S|}\int_S u \, dx$, S is any measurable subset of Ω.

$d = \operatorname{diam} \Omega$.

Proof. Assume $u \in C^1$, then use density.

$$u(x) - u(y) = -\int_0^{|x-y|} D_r(u)(x + r\omega) \, dr$$

$$\omega = \frac{y-x}{|y-x|} \in S^{n-1} \qquad x \xleftarrow{\omega} y$$

Integrate wrt y over S:

$$|S|(u(x)-u_S) \leq -\int_S \int_0^{|x-y|} D_r(u)(x+r\omega)\, dr\, dy$$

$$V(x) := \begin{cases} |D_r(u)(x)| & x \in \Omega \\ 0 & x \notin \Omega. \end{cases}$$

$$\Rightarrow |u(x) - u_S| \leq \frac{1}{|S|} \int_{|x-y|<\underset{\sim}{d}} \int_0^\infty V(x+r\omega)\, dr\, dy$$

diam $|S|$

$S \subseteq \{y \mid |x-y|<d\}$. over estimate.

$$= \frac{1}{|S|} \int_0^\infty \int_{|\omega|=1} \int_0^d V(x+r\omega) \rho^{n-1} d\rho\, d\omega\, dr$$

$\{|x-y|<d\}$ in polar coordinates

$$= \frac{d^n}{n|S|} \int_0^\infty \int_{|\omega|=1} V(x+r\omega)\, d\omega\, dr$$

$$= \frac{d^n}{n|S|} \int_\Omega |x-y|^{1-n} |D_r(u)(y)|\, dy \qquad \blacksquare$$

Thm (Morrey's Embedding) $u \in W_0^{1,p}(\Omega)$ $p>n$. then $u \in C^\gamma(\bar{\Omega})$ where $\gamma = 1-\frac{n}{p}$

Furthermore, $\forall$ ball B_R of radius R.

$$\underset{\Omega \cap B_R}{\operatorname{osc}}\, u \leq C R^\gamma \|Du\|_p . \quad C = C(n,p).$$

$$\underset{A}{\operatorname{osc}}\, u := \sup_{x,y \in A} |u(x)-u(y)|$$

Proof. In our previous Lemma, take $S = B := \Omega \cap B_R$

$$|u(x)-u_B| \lesssim_{n,p} R^\gamma \|Du\|_p \quad \text{a.e. } x \in B.$$

Then $|u(x)-u(y)| \lesssim_{n,p} R^\gamma \|Du\|_p$ a.e. $x, y \in B$. $\blacksquare$

a factor of 2

Remarks

1) $u \in W_0^{1,p}(\Omega)$ $p>n$. $\Rightarrow \|u\|_{C^\gamma(\bar{\Omega})} \lesssim (1+\operatorname{diam}(\Omega)^\gamma) \|Du\|_p$

$$= \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|}$$

2) $W_0^{1,p}(\Omega) \to L^{\frac{np}{n-p}}$ if $p<n$

$\to L^{\varphi}(\Omega)$ $\varphi(t) = e^{|t|^{\frac{n}{n-1}}} - 1$ if $p=n$

$\to C^{\gamma}(\bar{\Omega})$ $\gamma = 1-\frac{n}{p}$ if $p>n$

Thm (Morrey's Inequality) Evans

Assume $n<p\leq\infty$. Then

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \lesssim_{n,p} \|u\|_{W^{1,p}(\mathbb{R}^n)} \qquad \gamma := 1-\frac{n}{p}$$

$$\boxed{\forall\ u \in C^1(\mathbb{R}^n)}$$

Proof.

① $\frac{1}{|B|}\int_{B(x,r)} \cdots dy = ⨍_{B(x,r)} |u(y)-u(x)|\, dy \lesssim_n \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy \quad \forall\ B(x,r) \subseteq \mathbb{R}^n$

Proof of ①: for simplicity, assume $x=0$. $u(0)=0$. $r=1$ by rescaling. ① becomes

$$\int_{B(0,1)} |u(y)|\, dy \lesssim \int_{B(0,1)} \frac{|Du(y)|}{|y|^{n-1}}\, dy$$

It's enough to show $\int_{B(0,1)} |u(y)|\, dy \lesssim \int_0^1 \int_{\partial B(0,1)} |D(u)(t\omega)|\, d\omega\, dt$

It's true by fundamental theorem of calculus.

② $\sup_{\mathbb{R}^n} |u| \lesssim \|u\|_{W^{1,p}(\mathbb{R}^n)}$

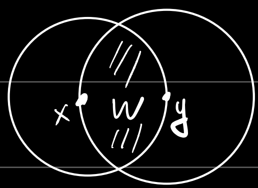
Proof. $|u(x)| \leq ⨍_{B(x,1)} |u(x)-u(y)|\, dy + ⨍_{B(x,1)} |u(y)|\, dy$

use ① $\lesssim_{n,p} \int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy + \|u\|_p$

Hölder
$$\lesssim \left(\int_{\mathbb{R}^n} \|Du\|_p \, dy\right)\left(\int_{B(x,1)} \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}} + \|u\|_p$$

$$\lesssim \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

③ For any $x, y \in \mathbb{R}^n$. choose $W = B(x,r) \cap B(y,r)$, $r := |x-y|$.



$$|u(x)-u(y)| \lesssim \fint_W |u(x)-u(z)| \, dz + \fint_W |u(y)-u(z)| \, dz$$

$$\lesssim \fint_{B(x,r)} |u(x)-u(z)| \, dz + \int_{B(y,r)} |u(y)-u(z)| \, dz \qquad \left(|W| \sim |B(x,r)| \sim r^n\right)$$

use ①
$$\lesssim \int_{B(x,r)} \frac{|Du(z)|}{|z-x|^{n-1}} \, dz + \int_{B(y,r)} \frac{|Du(z)|}{|z-y|^{n-1}} \, dz$$

Hölder
$$\lesssim \|Du\|_p \left\| \frac{1}{|\cdot - x|^{n-1}} \right\|_{p'}$$

$$\lesssim \|Du\|_p \, r^{1-\frac{n}{p}}$$

$$\Rightarrow |u(x)-u(y)| \lesssim |x-y|^{1-\frac{n}{p}} \|Du\|_p$$

$$\Rightarrow \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{\gamma}} \lesssim \|Du\|_p$$

$$\|u\|_{C^{0,\gamma}(\Omega)} \lesssim \|u\|_{W^{1,p}(\Omega)}$$

RMK. $|u(x)-u(y)| \lesssim r^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |Du(z)|^p \, dz\right)^{\frac{1}{p}}$

$\forall \; u \in C^1(B(x,2r))$ $y \in B(x,r)$ $n < p < \infty$

Hence also valid for $u \in W^{1,p}(B(x,2r))$.

Actually $|u(x)-u(y)| \lesssim r^{1-\frac{n}{p}} \left(\int_{B(x,r)} |Du(z)|^p \, dz\right)^{\frac{1}{p}}$

Def. We say u^* is a version of a given function u if $u \equiv u^*$ a.e.

Thm. (Estimates for $W^{1,p}$ $n<p\le\infty$)

Let $U \subseteq \mathbb{R}^n$ be open bounded and suppose ∂U is C^1 $n<p\le\infty$. $u \in W^{1,p}(U)$

For extension purpose.

Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ for $\gamma = 1-\frac{n}{p}$.

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \lesssim_{n,p,U} \|u\|_{W^{1,p}(U)}$$

Remark. Hence we will always identify u with Hölder continous version u^*.

Proof. Since ∂U is C^1, by extension Thm we can extend u to $\mathbb{R}^n$. $\bar{u} = Eu \in W^{1,p}(\mathbb{R}^n)$

$$\begin{cases} \bar{u} = u \text{ in } U \\ \bar{u} \text{ has comp. supp.} \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,p}(U)} \end{cases} \quad (*)$$

Now assume $n<p<\infty$. Since $\bar{u}$ has compact support, we can find $u_m \in C_c^\infty(\mathbb{R}^n) \longrightarrow \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$ $(**)$

Now apply Morrey's to u_m:

$$\|u_m - u_n\|_{C^{0,\gamma}(\mathbb{R}^n)} \lesssim \|u_m - u_n\|_{W^{1,p}(\mathbb{R}^n)}$$ Hence $\{u_m\}$ Cauchy in $C^{0,\gamma}(\mathbb{R}^n)$.

Banach.

Denote the limit by $u^* := \lim_{m\to\infty} u_m$ in $C^{0,\gamma}(\mathbb{R}^n)$ $(***)$

$\Longrightarrow$ $u = u^*$ a.e. on U, combing $(*)$, $(**)$ and $(***)$.

$$\|u^*\|_{C^{0,\gamma}} \lesssim \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,p}(U)}$$ ∎

*Thm (General Sobolev Inequalities) useful in PDE. can improve regularity if $kp > n$.

Let $U \subseteq \mathbb{R}^n$ bounded open with C^1 ∂U. Assume $u \in W^{k,p}(U)$. Then

(i) If $kp < n$: $u \in L^q(U)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$

$$\|u\|_{L^q(U)} \lesssim_{k,p,n,U} \|u\|_{W^{k,p}(U)}$$

(ii) If $kp > n$: $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\overline{U})$ where $\gamma = \begin{cases} [\frac{n}{p}]+1-\frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{N} \\ \text{any positive number} < 1 & \text{if } \frac{n}{p} \in \mathbb{N} \end{cases}$

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{U})} \lesssim_{k,p,n,U,\underset{\text{if } \frac{n}{p}\in\mathbb{N}}{\gamma}} \|u\|_{W^{k,p}(U)}$$

RMK We only assume ∂U to be C^1

Proof. (i) $k < \frac{n}{p}$: Since $D^\alpha u \in L^p(U)$ $\forall |\alpha| \le k$. by Gagliado - Nirenburg - Sobolev:

$$\|D^\beta u\|_{L^{p^*}(U)} \lesssim \|u\|_{W^{k,p}(U)} \quad \forall |\beta| \le k-1 \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

$\Rightarrow u \in W^{k-1,p^*}(U)$

$\Rightarrow u \in W^{k-2,(p^*)^*}(U)$ $\qquad \frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$

Repeating k steps: $u \in W^{0,q}(U)$. $\qquad$ Goal: $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.

(ii) $k > \frac{n}{p}$: ① Simple case: $\frac{n}{p}$ is NOT an integer. Apply similar argument as in (i)

$u \in W^{k-l,r}(U)$ $\frac{1}{r} = \frac{1}{p} - \frac{l}{n}$ by G-N-S l-times. provided that $lp < n$.

Now choose $l = [\frac{n}{p}]$ the biggest integer s.t. $lp < n$. : $l < \frac{n}{p} < l+1$

$$\begin{cases} \frac{1}{r} = \frac{1}{p} - \frac{l}{n} \\ l < \frac{n}{p} < l+1 \end{cases} \Rightarrow r = \frac{pn}{n-pl} > n. \qquad u \in W^{k-l,r}$$

Therefore, we could apply Morrey's Inequality:

$$D^\alpha u \in C^{0,\underbrace{1-\frac{n}{r}}}(U) \quad \forall |\alpha| \le k-l-1$$

$= 1 - \frac{n}{p} + l = [\frac{n}{p}] + 1 - \frac{n}{p} := \gamma$ as desired.

$\Rightarrow u \in C^{k-[\frac{n}{p}]-1,\gamma}$

② if $\frac{n}{p} \in \mathbb{N}$: $\quad l := [\frac{n}{p}] - 1 = \frac{n}{p} - 1$. $\quad u \in W^{k-l, r}(U)$ $\quad \frac{1}{r} = \frac{1}{p} - \frac{l}{n} \Leftrightarrow r = \frac{pn}{n-pl} = n$.

G-N-S inequality yields

$$D^{\alpha} u \in L^{q}(U), \quad \forall\ n \le q < \infty. \quad |\alpha| < k - l - 1 = k - \frac{n}{p}$$

By Morrey's inequality: $\quad D^{\alpha} u \in C^{0, 1-\frac{n}{q}}(\overline{U}) \quad \forall\ n < q < \infty. \quad |\alpha| \le k - \frac{n}{p} - 1$.

$\Rightarrow \quad u \in C^{k-\frac{n}{p}-1, \gamma}$ as desired. ■

Compactness of Embedding

G-N-S gives $\quad W^{1,p}(U) \hookrightarrow L^{p^*}(U)$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. $\quad 1 \le p < n$. $\quad U$ is open. bounded with C^1 ∂U.

This embedding is compact.

Def. Compact embedding. Let X, Y be Banach Spaces. $\quad X \subset Y$: $\|u\|_X \lesssim \|u\|_Y$

We say X is compactly embedded in Y $\left(X \subset\subset Y,\ X \overset{c}{\hookrightarrow} Y \right)$ if. $\forall$ bounded sequence $\{u_n\} \subset X$. is pre compact in Y, i.e. contains a convergent subsequence in Y. If $\|u_n\|_X < +\infty$ $\forall n$. then $\exists \{u_{n_j}\}$. $u \in Y$. $\|u_{n_j} - u\|_Y \longrightarrow 0$.

RMK. Id_X is a compact operator.

Thm. (Rellish - Kondrachov compactness theorem)

Assume $U \subseteq \mathbb{R}^n$. open. bounded, with C^1 boundary. Suppose $1 \le p < n$. then

$$W^{1,p}(U) \overset{c}{\hookrightarrow} L^{q}(U) \quad \forall\ q \in [1, p^*) \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Proof. Step 1. Fix $1 \le q < p^*$. Recall $W^{1,p} \hookrightarrow L^q(U)$ (proved before). $\|u\|_{L^q} \lesssim \|u\|_{W^{1,p}}$

Only need to check compactness: if u_m bounded sequence in $W^{1,p}$, then $\exists$ sub- u_{m_j} conv. in L^q.

For simplicity. since U is bounded with C^1 boundary. by extension theorem, we may WLOG assume:

- $\{u_m\}$ have compact support on $V \subseteq \mathbb{R}^n$.

- $u_m \in W^{1,p}(\mathbb{R}^n)$.
- $\sup_m \|u_m\|_{W^{1,p}(\mathbb{R}^n)} < +\infty$.

We work with mollifier: $u_m^\varepsilon := \eta_\varepsilon * u_m$. $\quad \mathrm{supp}\,(u_m^\varepsilon) \subseteq V$.

Claim: $u_m^\varepsilon \to u_m$ in $L^q(V)$ as $\varepsilon \to 0^+$ uniformly in m.

Proof of the claim: idea: pretend $u_m \in C^\infty$. $u_m^\varepsilon(x) - u_m(x) = \frac{1}{\varepsilon^n}\int \eta\left(\frac{x-z}{\varepsilon}\right)(u_m(z) - u_m(x))\,dz$

$= \cdots$

$= -\varepsilon \int \eta(y) \int_0^1 Du_m(x - \varepsilon t y)\cdot y \,dt\,dy$

$$\int_V |u_m^\varepsilon - u_m|\,dx \le \varepsilon \int_V |Du_m(z)|\,dz \Rightarrow \|u_m^\varepsilon - u_m\|_{L^1(V)} \le \varepsilon \|Du_m\|_{L^1(V)} \overset{V \text{ is bounded}}{\lesssim} \varepsilon \|Du_m\|_{L^p(V)}$$

General case: if u_m is not smooth, approximate it by smooth functions

$\Rightarrow u_m^\varepsilon \to u_m$ in $L^1(V)$ uniformly

Hölder

$\|u_m^\varepsilon - u_m\|_{L^q} \le \|u_m^\varepsilon - u_m\|_{L^1}^{\theta} \|u_m^\varepsilon - u_m\|_{L^{p^*}}^{1-\theta}$ where $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$ Done.

Claim: For each $\varepsilon > 0$, $\{u_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded and equi-continuous.

Proof. $\|u_m^\varepsilon\|_\infty \le \|u_m\|_{L^1} \|\eta_\varepsilon\|_\infty \lesssim \frac{1}{\varepsilon^n} < +\infty \quad \forall\, m$

$\|Du_m^\varepsilon\|_\infty \lesssim \frac{1}{\varepsilon^{n+1}} < +\infty$ Done.

By Arzela-Ascoli Thm. + Diagonal argument:

Fix $\delta > 0$. Claim: $\exists \{u_{m_j}\}_{j=1}^\infty$ $\limsup_{k,j\to\infty} \|u_{m_k} - u_{m_j}\|_{L^q(V)} \le \delta$.

Proof: Choose ε small s.t. $\|u_m^\varepsilon - u_m\|_{L^q} \lesssim \frac{\delta}{2}$.

Apply A-A to u_m^ε: $\limsup_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} = 0$.

$$\Rightarrow \limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta.$$

Now diagonal argument.

$\delta = 1$: $\boxed{u_{11}}$, u_{12}, u_{13}, u_{14}, u_{15}, u_{16} ----

$\delta = \frac{1}{2}$: u_{21} $\boxed{u_{22}}$ u_{23} u_{24} u_{25} u_{26} - -

$\delta = \frac{1}{4}$ u_{31} u_{32} $\boxed{u_{33}}$ u_{34} u_{35} u_{36} - - -

$\delta = \frac{1}{8}$ u_{41} u_{42} u_{43} $\boxed{u_{44}}$ - - - -