Take $\{u_{nn}\}_{n=1}^{\infty}$. u_{nn} converges in $L^q(V)$

RMK. $p^* > p$, $p^* \to \infty$ as $p \to n$.

$\Rightarrow W^{1,p}(U) \overset{c}{\hookrightarrow} L^p(U) \quad \forall\ p \in [1, \infty]$.

For $n < p \leq \infty$

Poincaré Inequality

$$(u)_U = \fint_U u\,dy = \frac{1}{|U|}\int_U u\,dy$$

Thm. U bounded, open, connected subset of $\mathbb{R}^n$, with C^1 boundary. $1 \le p \le \infty$.

$$\|u-(u)_U\|_{L^p(U)} \lesssim_{n,p,U} \|Du\|_{L^p(U)} \quad \forall\; u \in W^{1,p}$$

RMK. if $|U|<\infty$, then can control u in terms of its gradient.

Proof. We argue by contradiction. Suppose the ineq. is false: Fix $U \subseteq \mathbb{R}^n$. For each $k=1,2,\dots$ we can find

$u_k \in W^{1,p}(U)$ s.t. $\|u_k-(u_k)_U\|_{L^p} > k\|Du_k\|_{L^p}$

Re-normalize $v_k = \dfrac{u_k-(u_k)_U}{\|u_k-(u_k)_U\|_{L^p(U)}}$, $k=1,2,\dots$

$$\Rightarrow \begin{cases} \|v_k\|_{L^p}=1 \\ (v_k)_U = 0 \\ \|Dv_k\|_p \le \frac{1}{k} \end{cases} \Rightarrow v_k \text{ bounded in } W^{1,p}(U)$$

By using Rellich-Kondrachov (and the remarks after)

$\exists$ a subsequence $u_{k_j} \in W^{1,p}(U)$ converging in $L^p(U)$ to $v \in L^p(U)$.

$$\begin{cases} \|v_{k_j}\|_p = 1 \\ (v_{k_j})_U = 0 \end{cases} \Rightarrow \begin{cases} \|v\|_p = 1 \\ (v)_U = 0 \end{cases}$$

$$\|Dv_{k_j}\|_p \le \frac{1}{k_j} \overset{?}{\Longrightarrow} Dv = 0$$

Proof of the "$\overset{?}{\Rightarrow}$": Take $\phi \in C_c^\infty(U)$. By def of weak derivative:

$$-\int_U v\phi_{x_i} = -\lim_{j\to\infty}\int_U v_{k_j}\phi_{x_i} = \lim_{j\to\infty}\int_U (v_{k_j})_{x_i}\phi \lesssim \lim_{j\to\infty}\frac{1}{k_j} = 0$$

$\Rightarrow Dv = 0$.

Now $v \in W^{1,p}$, $Dv=0$ a.e. U open connected $\Rightarrow v \equiv$ Const. a.e.

But $\|v\|=1$. $(v)_U = 0$.

$\Downarrow$ v cannot be 0 a.e. $\quad$ $\Downarrow$ $v = $ Const. $=0$. Contradiction. $\blacksquare$

Thm. Poincaré Ineq. for a Ball

$$\| u-(u)_{B(x,r)} \|_{L^p(B(x,r))} \lesssim \underset{\triangle}{r} \cdot \|Du\|_{L^p(B(x,r))} \qquad \forall\, u \in W^{1,p}(B(x,r))$$

Proof. Case $U = B(0,1)$ follows from the previous thm.

General case: Rescale $v(y) = u(x+ry) \quad y \in B(0,1)$

Then $v \in W^{1,p}(B(0,1)) \qquad \| v-(v)_{B(0,1)} \|_{L^p(B(0,1))} \lesssim \|Dv\|_{L^p(B(0,1))}$

Now change of variable. RHS will yield a "r" factor. ∎

RMK. Connection of BMO and $W^{1,n}$

Assume $u \in W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ $(n \geq 2)$ Take $p=1$ in the previous theorem.

$$\fint_{B(x,r)} u-(u)_{B(x,r)} \lesssim r \fint_{B(x,r)} |Du| \cdot 1$$

$$\overset{\text{Hölder}}{\lesssim} r \left(\fint_{B(x,r)} |Du|^n \right)^{\frac{1}{n}}$$

$$= \frac{1}{(\alpha(n))^{\frac{1}{n}}} \left(\int |Du|^n \right)^{\frac{1}{n}}$$

$$\Rightarrow \quad u \in \boxed{BMO(\mathbb{R}^n)}$$

Def. $[u]_{BMO(\mathbb{R}^n)} := \sup_{B(x,r) \subseteq \mathbb{R}^n} \left\{ \fint_{B(x,r)} |u-(u)_{B(x,r)}| \right\}$.

Difference quotients. 差商

Assume $u: U \longrightarrow \mathbb{R}^n \in L^1_{loc}(U)$. $V \subset\subset U$.

Def. (i) The i-th difference quotient of size h

$$D_i^h u(x) := \frac{u(x+he_i) - u(x)}{h} \quad (i=1,2,\cdots,n) \quad \forall x \in V. \quad h \in \mathbb{R}. \quad 0<|h|< \operatorname{dist}(V, \partial U)$$

(ii) $D^h u := (D_1^h u, \cdots, D_n^h u)$

Theorem

(i) $1 \le p < \infty$. $u \in W^{1,p}(U)$ For each $V \subset\subset U$, we have $\|D^h u\|_{L^p(V)} \lesssim_{U,V,p,n} \|Du\|_{L^p(U)}$

for each $0<|h|<\frac{1}{2}\operatorname{dist}(V, \partial U)$

Proof of (i): Assume $u \in C^\infty(V)$ then use density.

$$u(x+he_i) - u(x) = h\int_0^1 u_{x_i}(x+the_i)\,dt$$

$$|u(x+he_i) - u(x)| \lesssim |h| \int_0^1 |Du(x+hte_i)|\,dt$$

$$\Rightarrow \int_V |D^h u|^p dx \lesssim \sum_{i=1}^n \int_V \int_0^1 |Du(x+hte_i)|^p\,dt\,dx$$

$$= \sum_{i=1}^n \int_0^1 \int_V |Du(x+hte_i)|^p\,dx\,dt$$

$$\lesssim \int_U |Du|^p dx \int_0^1 dt$$

$$= \|Du\|^p_{L^p(U)}.$$

(ii) Assume $1<p<\infty$ $u \in L^p(V)$ $\exists$ const. C s.t.

$$\|D^h u\|_{L^p} \le C. \quad \forall\ 0<|h|<\frac{1}{2}\operatorname{dist}(V, \partial U)$$

Then: $u \in W^{1,p}(V)$. $\|Du\|_{L^p(V)} \le C$.

Proof. Suppose $\|D^h u\|_{L^p(V)} \le C$ $\forall\ 0<h<\frac{1}{2}\operatorname{dist}(V, \partial U)$. Choose $\phi \in C_c^\infty(V)$ then for small h

$$\int_V u(x)\left[\frac{\phi(x+he_i) - \phi(x)}{h}\right] dx = -\int_V \frac{u(x) - u(x-he_i)}{h}\phi(x)\,dx$$

$$\boxed{\int_V u(D_i^h \phi) = -\int_V (D_i^{-h} u)\phi}$$

Integration by parts formula.

$\sup_h \|D_i^{-h} u\|_{L^p(V)} < \infty$ Behave like Balzano - Wierestrass

Now since $1 < p < \infty$, there exists a function $v_i \in L^p(V)$ and $h_k \to 0$ s.t.

$$D_i^{h_k} u \longrightarrow v_i \quad \text{weakly in } L^p(V)$$

$$\int_V u \phi_{x_i} = \int_U u \phi_{x_i} = \lim_{h_k \to 0} \int_U u \, D_i^{h} \phi$$

$$\overset{\text{integration by parts}}{=} - \lim_{h_k \to 0} \int_U (D_i^{-h_k} u) \phi$$

$$\overset{\text{weak conv.}}{=} - \int_V v_i \phi = - \int_U v_i \phi$$

$\Rightarrow v_i = u_{x_i}$ in the weak sense

$$\Rightarrow \begin{cases} Du \in L^p(V) \\ u \in L^p(V) \end{cases} \Rightarrow u \in W^{1,p}(V)$$

*Theorem. (Characterization of $W^{1,\infty}$)

Let $U \subseteq \mathbb{R}^n$ be open bounded with C^1 boundary. Then

$u: U \longrightarrow \mathbb{R}$ is Lipschitz continuous $\iff u \in W^{1,\infty}(U)$

Proof. 1. First assume $U = \mathbb{R}^n$, and u is compact supported.

① Suppose $u \in W^{1,\infty}(\mathbb{R}^n)$. $\|Du\|_\infty, \|u\|_\infty < +\infty$. Take $u^\varepsilon = \eta_\varepsilon * u$ the mollification of u. Then $u^\varepsilon \in C_c^\infty(\mathbb{R}^n)$

$u^\varepsilon \to u$ unif. as $\varepsilon \to 0$ on finite domain. $u \in W^{1,p}$ ($p \geq n$). use Morrey. u is cts

Take $x, y \in \mathbb{R}^n$. $x \neq y$.
$$u^\varepsilon(x) - u^\varepsilon(y) = \int_0^1 \frac{d}{dt} u^\varepsilon(tx + (1-t)y)\, dt$$ (Fund. Thm. of Calc.) Does not hold for original function u.

$$= \int_0^1 (Du^\varepsilon)(tx + (1-t)y)\, dt \cdot (x-y)$$

$$\Rightarrow |u^\varepsilon(x) - u^\varepsilon(y)| \leq \|Du^\varepsilon\|_\infty |x-y|$$

$$\leq \|Du\|_\infty |x-y|.$$

Now send $\varepsilon \to 0$. use unif. convergence. we are done.

② Suppose u is Lipschitz continuous: use Diff. quotient.

u Lipchitz $\Rightarrow \|D_i^{-h} u\|_\infty \le$ Lipchitz const. of $u := \mathrm{Lip}(u)$

there exists a function $v_i \in L^\infty(\mathbb{R}^n)$ and a subsequence $h_k \to 0$ s.t. $D_i^{-h_k} u \xrightarrow{\text{weak}} v_i$ in $L^2_{loc}(\mathbb{R}^n)$

$$\Rightarrow \int_{\mathbb{R}^n} \phi_{x_i} \cdot u = \lim_{h_k \to 0} \int_{\mathbb{R}^n} u \cdot D_i^{h_k} \phi = \lim_{h_k \to 0} - \int_{\mathbb{R}^n} \left(D_i^{-h_k} u\right) \phi$$

$$= - \int_{\mathbb{R}^n} v_i \phi$$

$\Rightarrow$ $v_i = u_{x_i}$ in the weak sense, $i = 1, 2, \cdots, n$.

$\Rightarrow$ $u \in W^{1,\infty}(\mathbb{R}^n)$

RMK. the trick of Diff. quotient.

Classical: $D_i^h u := \dfrac{u(x+he_i) - u(x)}{h} \to D_i u$

Sobolev: $D_i^h u \xrightarrow{\text{weakly}} D_i u$ weak derivative

2. General Case: U bounded. ∂U is C^1.

We first extend u to $\bar{u} := Eu$ on $\mathbb{R}^n$. Repeat the same argument. ∎

RMK. For any open set U, also prove: u is locally Lipschitz $\Longleftrightarrow$ $u \in W^{1,\infty}_{loc}(U)$

If $n < p < \infty$: $u \in W^{1,p} \xRightarrow{\text{Morrey}} u \in C^{0, 1-\frac{n}{p}}$

Differentiability a.e.

Def. A function $u: U \to \mathbb{R}$ is differentiable at $x_0 \in U$ if $\exists\ a \in \mathbb{R}^n$ s.t.

$$u(y) = u(x) + a\cdot(y-x) + o(|y-x|)$$

i.e. $\lim_{y\to x} \frac{|u(y)-u(x)-a\cdot(y-x)|}{|y-x|} = 0.$

RMK. a is unique. Denote $Du := a$.

Theorem. (Differentiability a.e.)

Assume $u \in W^{1,p}_{loc}(U)$ for some $n<p\le\infty$. Then u is differentiable a.e. in U.

It's gradient equals its weak gradient a.e.

RMK. We identify u as its continuous version, as before.

Proof Recall Morrey for $n<p<\infty$: $\left|v(y)-v(x)\right| \lesssim r^{1-\frac{n}{p}}\left(\int_{B(x,2r)} |Dv(z)|^p\,dz\right)^{\frac{1}{p}}$, $y\in B(x,r)$

valid for v $v\in C^1$. (By approximation, it's also true for $u \in W^{1,p}$).

If one has $\int_{B(x,2r)} |Dv(x)|^p\,dz$, Then

$|v(x)-v(y)| \lesssim r^{1-\frac{n}{p}}\,|Dv(x)|\,r^{\frac{n}{p}}$

$\lesssim |Dv(x)|\,r$. as needed.

Hence Hope: $\int_{B(x,2r)} |Dv(x)-Dv(z)|^p\,dz$ small.

Formal proof: Choose $u\in W^{1,p}_{loc}(U)$ Then for Lebesgue pts $x\in U$, by Lebesgue Diff. Thm.

we have $⨍_{B(x,r)} |Du(z)-Du(x)|^p \longrightarrow 0$ as $r\to 0$.

Consider $v(y) := u(y)-u(x)-Du(x)\cdot(y-x)$

$$\Rightarrow |u(y)-u(x)-Du(x)\cdot(y-x)| \lesssim r^{1-\frac{n}{p}}\left(\int_{B(x,2r)} |Du(x)-Du(z)|^p\,dz\right)^{\frac{1}{p}}$$

$$\lesssim r^{1-\frac{n}{p}}\cdot r^{\frac{n}{p}}\left(⨍_{B(x,2r)} |Du(x)-Du(z)|\,dz\right)^{\frac{1}{p}}$$

$\longrightarrow 0$ as $r \to 0^+$

Hence u is differentiable at x.

Now if $p = \infty$. note that $W^{1,\infty}_{loc} \subseteq W^{1,p}_{loc}$ $\forall\, n < p < \infty$. ■

Corollary (Rademacher's Diff. Thm) u is locally Lipschitz continuous on U. then u is differentiable a.e. on U.

RMK. The usual proof of Rademacher's Diff. Thm goes in two steps.

- Step 1 : 1 dimension case.
- 2 : n dimension

Hardy's inequality

Thm. $(n \geq 3)$ Suppose $u \in H^1(B(0,r)) := W^{1,2}(B(0,r))$. Then $\frac{u}{|x|} \in L^2(B(0,r))$

$$\int_{B(0,r)} \frac{u^2}{|x|^2}\,dx \lesssim \int_{B(0,r)} |Du|^2 + \frac{u^2}{r^2}\,dx.$$

Proof. Assume $r = 1$. $u \in C^\infty$.

Note $D\left(\frac{1}{|x|}\right) = -\frac{x}{|x|^3}$.

$$\int_{B(0,1)} \frac{u^2}{|x|^2}\,dx = -\int_{B(0,1)} u^2 D\left(\frac{1}{|x|}\right)\cdot\frac{x}{|x|}\,dx$$

$$= \int_{B(0,1)} \left[2u Du \cdot \frac{x}{|x|^2} + (n-1)\frac{u^2}{|x|^2}\right] dx - \int_{\partial B(0,1)} u^2 \nu \cdot \underbrace{\frac{x}{|x|^2}}_{|x|=1 \text{ on } \partial B}\,dS$$

$$\Rightarrow (2-n)\int \frac{u^2}{|x|^2}\,dx = 2\int \underbrace{u Du \cdot \frac{x}{|x|^2}}_{\text{Cauchy Swartz}}\,dx - \underbrace{\int_{\partial B(0,1)} u^2\,dS}$$

$$= \int \operatorname{div}(x u^2)\,dx$$

$$= \int_{B(0,1)} n u^2 + 2u Du \cdot$$

Thm. Characterization of H^s by Fourier transf.

Suppose $k \in \mathbb{N}$. (i) A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ iff $(1+|\xi|^k)\hat{u}(\xi) \in L^2(\mathbb{R}^n)$

(ii) For each $u \in H^k(\mathbb{R}^n)$, $\frac{1}{C_{k,n}}\|u\|_{H^k(\mathbb{R}^n)} \le \|(1+|\xi|^k)\hat{u}(\xi)\|_{L^2} \le C_{k,n}\|u\|_{H^k(\mathbb{R}^n)}$

Proof. 1. Assume $u \in H^k(\mathbb{R}^n) \Rightarrow D^\alpha u \in L^2(\mathbb{R}^n)$ $\forall |\alpha| \le k$.

Convention of Fourier Transf:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$$

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi\cdot x}\,d\xi.$$

Suppose $u \in C_c^k(\mathbb{R}^n)$. Then we have $\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$

$$\Rightarrow |\xi|^\alpha \hat{u}(\xi) \in L^2 \quad \forall |\alpha| \le k.$$

Choose $\alpha = (k,0,\cdots,0), (0,k,0,\cdots,0), \cdots, (0,\cdots,0,k)$.

$$\Rightarrow \int_{\mathbb{R}^n} (1+|\xi|^k)^2 \hat{u}(\xi)\,d\xi \le C\|u\|_{H^k}.$$

By density, we could remove the assumption of $u \in C_c^k(\mathbb{R}^n)$

2. Conversely, if $(1+|\xi|^k)\hat{u} \in L^2(\mathbb{R}^n)$:

$$\|\underbrace{(i\xi)^\alpha \hat{u}}_{=\widehat{D^\alpha u}}\|_{L^2}^2 \le \int |\xi|^{2|\alpha|} |\hat{u}|\,d\xi \lesssim \|(1+|\xi|^k)\hat{u}\|_{L^2}^2$$

$$\Rightarrow \widehat{D^\alpha u} \in L^2 \overset{\text{Plancherel}}{\Leftrightarrow} D^\alpha u \in L^2. \Longrightarrow u \in H^k(\mathbb{R}^n) \qquad \square$$

Definition. Assume $s \in \mathbb{R}$. $u \in L^2(\mathbb{R}^n)$ We define

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1+|\xi|^s)\hat{u}(\xi)\|_{L^2(\mathbb{R}^n)} \quad \text{if it's finite.}$$

The Space of H^{-1} as the dual of $H_0^1(U)$

Def. We denote $H^{-1}(U)$ the dual space of $H_0^1(U) := W_0^{1,2}(U)$

i.e. $f \in H^{-1}(U)$ if f is a bounded linear functional on $H_0^1(U)$.

RMK. We do not identify $H_0^1(U)$ with its dual,

Indeed, $H_0^1(U) \subset L^2 \subset H^{-1}(U)$

Notation: $\langle f, g \rangle$ pairing of $H^{-1}(U)$ and $H_0^1(U)$

Def. For $f \in H^{-1}(U)$, we define the norm

$$\|f\|_{H^{-1}(U)} := \sup\left\{ \langle f, g\rangle \;\middle|\; g \in H_0^1(U),\ \|g\|_{H_0^1(U)} \le 1 \right\}.$$

Theorem (Characterization of H^{-1}) Assume $f \in H^{-1}(U)$.

(i) $\exists$ functions $f^0, f^1, \dots, f^n$ in $L^2(U)$ s.t.

$$\langle f, v\rangle = \int_U f^0 v + \sum_{i=1}^n f^i \partial_{x_i} v \, dx \qquad (*) \qquad \forall v \in H_0^1(U)$$

(ii) $$\|f\|_{H^{-1}(U)} = \inf\left\{ \left(\int_U \sum_{i=0}^n |f^i|^2 dx \right)^{\frac{1}{2}} \right\}$$

(iii) $\langle v, u\rangle_{L^2(U)} = \langle v, u\rangle_{H^{-1}} \quad \forall u \in H_0^1(U),\ v \in L^2(U) \subset H^{-1}(U)$

RMK. When $(*)$ holds, we often write $f = f^0 - \sum_{i=1}^n \partial_{x_i} f^i = \underbrace{L^2 + \operatorname{div}}_{\text{function structure.}}(L^2\text{-vector function})$

Proof. (key idea: Riesz representation)

Given $u, v \in H_0^1(U)$, define $(u, v) := \int_U (Du \cdot Dv + uv)\, dx$, the inner product on $H_0^1(U)$

Let $f \in H^{-1}(U)$. By Reisz, we can find a unique representative $u \in H_0^1(U)$ s.t. $\langle f, v\rangle = (u, v)$ $\forall v \in H_0^1(U)$

$$\int_U Du \cdot Dv + uv \, dx = \langle f, v\rangle \quad \forall v \in H_0^1(U)$$

$\Rightarrow \begin{cases} f^0 = u \\ f^i = \partial_{x_i} u \end{cases}$ $(*)$ is then proved.

let $u = v$.

(ii). Assume $f \in H^{-1}(U)$ $\langle f, v\rangle = \int_U g^0 v + \sum g^i \partial_{x_i} v \, dx$ where $g^0, g^1, \dots, g^n \in L^2(U)$ is another characterization of f.

$$\int_U |Du|^2 + u^2 \, dx = (u, u) \le \int_U \sum_{i=0}^n |g^i|^2 dx$$

$$\Rightarrow \int_U \sum |f^i|^2 dx \le \int_U \sum |g^i|^2 dx \quad \text{since } f^0 = u,\ f^i = \partial_{x_i} u.$$

The characterization from Riesz is the smallest one

3. $|\langle f, v\rangle| \le \left(\int_U \sum |f^i|^2 dx\right)^{\frac{1}{2}} \|v\|_{H_0^1(U)}$

$\Rightarrow \|f\|_{H_0^1(U)} \le \left(\int_U \sum |f^i|^2 dx\right)^{\frac{1}{2}}$

The equality holds if $v = \dfrac{u}{\|u\|_{H_0^1(U)}}$ $\leftarrow$ u is the representive of f in $H_0^1(U)$ by Riesz.

$\square$

Elliptic Eqs.

$$\begin{cases} Lu = f & \text{in } U. \\ u = 0 & \text{on } \partial U \end{cases}$$

U is open bounded. f is given $Lu = -\sum \left(a^{ij}(x) u_{x_i}\right)_{x_j} + \sum b^i(x) u_{x_i} + c(x) u$ (Divergence form)

OR

$Lu = -\sum a^{ij}(x) u_{x_i} u_{x_j} + \sum b^i(x) u_i(x) + c(x) u.$ (Non-div. form)

Assume the symmetry condition: $a^{ij} = a^{ji}$.

Def. We say L is (uniformly) Elliptic if $\exists\ \theta > 0$. s.t. $\sum a^{ij}(x) \xi_i \xi_j \ge \theta |\xi|^2 \ \forall \xi \in \mathbb{R}^n$, a.e. $x \in U$.

In particular, $\left(a^{ij}(x)\right)_{i,j}$ is positive-defined, eigenvalues $\lambda_i \ge \theta$