We first introduce some important objects.

Tangent and Cotangent Bundle.

Suppose M is a n dimensional smooth manifold. $p \in M$. The **Tangent Bundle** of M, called "TM", is the disjoint union

$TM = \bigcup_{p \in M} T_pM = \{(p, v) : p \in M,\ v \in T_pM\}$. We could make TM into a manifold in the following way:

Let $F: U \subseteq \mathbb{R}^n \longrightarrow M$ be a local chart of M. Then F naturally induces a local chart $\tilde{F}: U \times \mathbb{R}^n \longrightarrow TM$ for TM:

$$\tilde{F}(u_1, \cdots, u_n, v^1, \cdots, v^n) = \left(F(u_1, \cdots, u_n), \sum_i v^i \frac{\partial}{\partial u_i}\right) \quad \text{where} \quad v = \sum_i v^i \frac{\partial}{\partial u_i} \in T_pM \quad p = F(u_1, \cdots, u_n)$$

RMK. One can show 2 local parametrizations are competiable with each other hence it's really a smooth structure on TM.

Similarly, the **cotangent bundle** $T^*M = \{(p, \alpha) : p \in M,\ \alpha \in T_p^*M\}$ can be parametrized by

$$\tilde{F}^*(u_1, \cdots, u_n, \alpha_1, \cdots, \alpha_n) = \Big(\underbrace{F(u_1, \cdots, u_n)}_{p \in M}, \underbrace{\sum_i \alpha_i\, du^i}_{\alpha \in T_p^*M}\Big) \quad \text{where } du^i \in T_p^*M \quad du^i\left(\frac{\partial}{\partial u_j}\right) = \delta^i_j$$

RMK. One can show 2 local parametrizations are competiable with each other hence it's really a smooth structure on T^*M.

(Smooth) Vector field. (0, 1) tensor.

A smooth vector field is a smooth map $V: M \to TM$. while we view TM as a smooth manifold as described above. i.e.

$\tilde{F}^{-1} \circ V \circ F: U \to U \times \mathbb{R}^n$ is smooth. In local coordinates, $V(p) = \left(p, \sum_i V^i(p) \frac{\partial}{\partial u_i}\right)$ $V^i: M \to \mathbb{R}$ are components of $V(p)$.

$$\tilde{F}^{-1} \circ V \circ F(u_1, \cdots, u_n) = \tilde{F}^{-1} V(\underbrace{F(u_1, \cdots, u_n)}_{p}) = \tilde{F}^{-1}\left(p, \sum_i V^i(p) \frac{\partial}{\partial u_i}\right) = \left(u_1, \cdots, u_n, V^1(u_1, \cdots, u_n), \cdots, V^n(u_1, \cdots, u_n)\right)$$

hence. **Prop.** A vector field V is smooth iff its components $\{V^i: M \to \mathbb{R}\}$ are smooth functions over M.

Differential 1-form. dual of "vector field", "cotangent field". (1, 0) tensor

A differential 1-form is a smooth map $\omega: M \to T^*M$ given by $\omega(p) = \left(p, \sum_i \alpha_i(p)\, du^i\right)$ in local coordinate $(u_1, \cdots, u_n)$.

Prop. A 1-form ω is smooth iff its components $\{\alpha_i: M \to \mathbb{R}\}$ are smooth functions over M.

Differential forms. multilinear alternating tensors.

0-form: smooth functions. $f = f(u_1, \cdots, u_n)$ in terms of local coordinates

1-form: cotangent vectors. $\omega(p) = \left(p, \sum_{i=1}^n \underbrace{\alpha_i(p)}_{\in C^\infty(M)}\, du^i|_p\right)$ in terms of local coordinates $\omega = \sum_{i=1}^n \alpha_i(u_1, \cdots, u_n)\, du^i$

k-form: constructed using wedge product "$\wedge$" as shown in the next section.

Next we introduce some operations on those objects.

Wedge product and higher forms.

Let $\alpha, \beta \in T^*M$. We want to construct a 2-form (bilinear alternating map). So define $\alpha\wedge\beta := \alpha\otimes\beta - \beta\otimes\alpha \in \underbrace{T^*M\otimes T^*M}_{(2,0)\text{ tensor}}$

3-forms: $\alpha\wedge\beta\wedge\gamma = \alpha\otimes\beta\otimes\gamma - \alpha\otimes\gamma\otimes\beta + \gamma\otimes\alpha\otimes\beta - \gamma\otimes\beta\otimes\alpha + \beta\otimes\gamma\otimes\alpha - \beta\otimes\alpha\otimes\gamma = \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\, \sigma(\alpha)\otimes\sigma(\beta)\otimes\sigma(\gamma)$.

$\operatorname{sgn}(\sigma) = +1$ if σ is even, -1 if σ is odd

k-forms: Let $\omega_1, \cdots, \omega_k \in T^*M$ then $\omega_1\wedge\cdots\wedge\omega_k = \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\, \sigma(\omega_1)\otimes\cdots\otimes\sigma(\omega_k)$. (not important)

In general a k-form looks like $\omega = \sum_{i_1,\cdots,i_k} \underbrace{\omega_{i_1\cdots i_k}}_{\in C^\infty(M)} dx_{i_1}\wedge\cdots\wedge dx_{i_k}$

Important features: **Proposition.**

1. $\omega\wedge\omega = 0$ if $\omega\in T^*M$ (Warning: if $\omega\in\Lambda^k T^*M$, $k>1$ $\omega\wedge\omega\neq 0$ in general.)
2. $T_1\wedge\cdots\wedge T_k = -T_{r(1)}\wedge\cdots\wedge T_{r(k)}$, $T_i\in T^*M$ if r is a transposition. i.e. exchange the order of any pair, we get a negative sign.
3. $\dim \Lambda^k T^*M = \binom{n}{k}$ where $\dim M = n$.
4. $\sigma\in\Lambda^k T^*M$, $\eta\in\Lambda^r T^*M$ then $\sigma\wedge\eta = (-1)^{kr}\eta\wedge\sigma$. In particular, even-form commutes with any other forms.
5. $(T_1\wedge\cdots\wedge T_k)\wedge(S_1\wedge\cdots\wedge S_r) = T_1\wedge\cdots\wedge T_k\wedge S_1\wedge\cdots\wedge S_r$
6. linearity: $(\alpha T + \beta S)\wedge\sigma = \alpha T\wedge\sigma + \beta S\wedge\sigma$. $T, S\in\Lambda^k T^*M$. $\alpha,\beta\in\mathbb{R}$. $\sigma\in\Lambda^s T^*M$
7. If T^*M is of dimension n. $\omega_i = \sum_{j=1}^n a_{ij}\, du^j \in T^*M$ then $\omega_1\wedge\cdots\wedge\omega_n = \det(a_{ij})\, du_1\wedge\cdots\wedge du_n$.

Examples $\omega = e^1\wedge e^2 + e^3\wedge e^4$. then $\omega\wedge\omega = (e^1\wedge e^2 + e^3\wedge e^4)\wedge(e^1\wedge e^2 + e^3\wedge e^4) = e^1\wedge e^2\wedge e^3\wedge e^4 + e^3\wedge e^4\wedge e^1\wedge e^2 = 2\, e^1\wedge e^2\wedge e^3\wedge e^4$.

remains

$=0$. since $e^i\wedge e^i = 0$

Corollary If $(u_1,\cdots,u_n)$ and $(v_1,\cdots,v_n)$ are two local charts of a manifold M. then

$$du^1\wedge\cdots\wedge du^n = \det\left(\frac{\partial(u_1,\cdots,u_n)}{\partial(v_1,\cdots,v_n)}\right) dv^1\cdots dv^n$$

Proof: Recall $du^i = \frac{\partial u_i}{\partial v_j} dv^j$ (with Einstein summation rule). then apply (7) above ∎

For simplicity we will apply the Einstein summation convention from now on.

$$a_i b^i = \sum_{i=1}^n a_i b^i.$$

* Push-foward vs Pull-back

Let $\Phi: M \to N$ be a smooth map. tangent map $\Phi_*: TM \to TN$.

Pull-back of a function: $f \in C^\infty(N)$. Then $\Phi^* f := f \circ \Phi : M \to \mathbb{R}$ is a smooth function on M.

Pull-back of 1-form

The pullback of 1-form $\Phi^*: T^*N \to T^*M$ is defined as:

$$\big(\underbrace{\Phi^* \underbrace{\omega}_{\in T^*N}}_{\in T^*M}\big)(\underbrace{V}_{\in TM}) = \boxed{\overset{\in\mathbb{R}}{\underbrace{\omega}_{\in T^*N}(\underbrace{\Phi_* V}_{\in TN})}} \text{ for all vector } V \in TM.$$

In terms of local coordinates: $M:(u_1,\cdots,u_m)$ $N:(v_1,\cdots,v_n)$ Recall $\Phi_*\left(\frac{\partial}{\partial u_j}\right) = \frac{\partial v_k}{\partial u_j}\frac{\partial}{\partial v_k}$, $[\Phi_*] = \left[\frac{\partial v_i}{\partial u_j}\right] = \frac{\partial(v_1,\cdots,v_n)}{\partial(u_1,\cdots,u_m)}$

$$(\Phi^* dv^i)\left(\frac{\partial}{\partial u_j}\right) = dv^i\left(\Phi_*\frac{\partial}{\partial u_j}\right) = dv^i\left(\sum_k \frac{\partial v_k}{\partial u_j}\frac{\partial}{\partial v_k}\right) = \sum_k \frac{\partial v_k}{\partial u_j}\underbrace{dv^i\left(\frac{\partial}{\partial v_k}\right)}_{\delta^i_k} = \frac{\partial v_i}{\partial u_j}$$

$\Rightarrow \Phi^* dv^i = \frac{\partial v^i}{\partial u_j}du_j$. $[\Phi^*] = \Big[\underbrace{\frac{\partial v^j}{\partial u^i}}_{\text{the }(i,j)\text{-entry}}\Big]$ wrt basis are "du^i" and "dv^j" for T^*M and T^*N respectively. $[\Phi^*] = [\Phi_*]^T$.

Pull-back of a tensor (hence of a diff. form).

Let $T := \underbrace{T_1 \otimes \cdots \otimes T_k}_{\text{1-forms (1,0) tensor}}$ be a $(k,0)$ Tensor on N. then define $(\Phi^*T)_p(X_1,\cdots,X_k) := T_{\Phi(p)}(\Phi_*X_1,\cdots,\Phi_*X_k)$ $\forall X_1,\cdots,X_k \in T_pM$

i.e. $\Phi^*(T_1\otimes\cdots\otimes T_k) = \Phi^*T_1 \otimes \cdots \otimes \Phi^*T_k$.

Eg. An example on $\mathbb{R}^2$. Let $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$ be $\Phi(x_1,x_2) = (\underbrace{e^{x_1+x_2}}_{y_1}, \underbrace{\sin(x_1^2x_2)}_{y_2}, \underbrace{x_1}_{y_3})$ Then $[\Phi_*] = \begin{bmatrix} e^{x_1+x_2} & e^{x_1+x_2} \\ 2x_2x_1\cos(x_1^2x_2) & x_1^2\cos(x_1^2x_2) \\ 1 & 0 \end{bmatrix}$

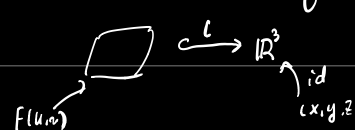
$[\Phi^*] = \begin{bmatrix} e^{x_1+x_2} & 2x_1x_2\cos(x_1^2x_2) & 1 \\ e^{x_1+x_2} & x_1^2\cos(x_1^2x_2) & 0 \end{bmatrix}$. $\Rightarrow \Phi^*(dy_1) = e^{x_1+x_2}dx_1 + e^{x_1+x_2}dx_2$ etc.

Let $f(y_1,y_2,y_3): \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. $T := \overset{(0,0)\text{ tensor}}{f(y_1,y_2,y_3)}\, dy_1 \otimes dy_2$ be a $(2,0)$ tensor. The pull-back of T under Φ is: $\Phi^*(T) = \Phi^*(f)\ \Phi^*(dy_1)\otimes\Phi^*(dy_2) = f(\Phi(x_1,x_2))\,e^{x_1+x_2}(dx_1+dx_2)\otimes\left(2x_1x_2\cos(x_1^2x_2)dx_1 + x_1^2\cos(x_1^2x_2)dx_2\right)$

$$= f\circ\Phi(x_1,x_2)\,e^{x_1+x_2}\cos(x_1^2x_2)\cdot x_1x_2\,(dx_1+dx_2)\otimes(2dx_1 + x_1dx_2)$$

$$= f\circ\Phi(x_1,x_2)\,e^{x_1+x_2}\cos(x_1^2x_2)\cdot x_1x_2\left(2dx_1^2 + 2dx_2dx_1 + x_1dx_1dx_2 + x_1dx_2^2\right).$$

Important Eg. Let $M \subseteq \mathbb{R}^3$ be a regular surface. $\iota: M \to \mathbb{R}^3$ be the inclusion map. $\underbrace{g_{\mathbb{R}^3} := dx^2+dy^2+dz^2}_{(2,0)\text{ tensor}}$ be the Euclidean metric on $\mathbb{R}^3$.



Now compute $\iota^* g$:

$id^{-1}\circ\iota\circ F(u,v) = (x(u,v), y(u,v), z(u,v))$. $[\iota_*] = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$. $[\iota^*] = \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix} \Rightarrow \iota^*(dx) = x_u du + x_v dv$.

Warning: do NOT commute.

$\iota^*(dx^2) = \iota^*(dx\otimes dx) = (\iota^* dx)\otimes(\iota^* dx) = (x_u du + x_v dv)\otimes(x_u du + x_v dv) = (x_u)^2du^2 + x_ux_v\,du\otimes dv + x_vx_u\,dv\otimes du + (x_v)^2dv^2$

Similarly, we have $\iota^*(dy^2) = y_u^2du^2 + y_uy_v\,dudv + y_vy_u\,dvdu + y_v^2dv^2$

$$\iota^*(dz^2) = z_u^2 du^2 + z_u z_v du dv + z_v z_u dv du + z_v^2 dv^2$$

Hence

$$\iota^*(g) = \iota^*(dx^2 + dy^2 + dz^2) \overset{\text{linearity}}{=} \iota^*(dx^2) + \iota^*(dy^2) + \iota^*(dz^2) = \begin{matrix} \boxed{(x_u)^2 du^2} + & \boxed{x_u x_v\, du \otimes dv} + & \boxed{x_v x_u\, dv \otimes du} + & \boxed{(x_v)^2 dv^2} \\ & & + & \\ \boxed{y_u^2 du^2} + & \boxed{y_u y_v\, du dv} + & \boxed{y_v y_u\, dv du} + & \boxed{y_v^2 dv^2} \\ & + & & \\ \boxed{z_u^2 du^2} + & \boxed{z_u z_v\, du dv} + & \boxed{z_v z_u\, dv du} + & \boxed{z_v^2 dv^2} \end{matrix}$$

$$= (x_u^2 + y_u^2 + z_u^2) du^2 + (x_u x_v + y_u y_v + z_u z_v) du dv + (x_v x_u + y_v y_u + z_v z_u) dv du + (x_v^2 + y_v^2 + z_v^2) dv^2$$

exactly the first fundamental form of M. ∎

Pull-back of a differential form. $\omega \in \Lambda^k T^* N$ a k-form on N. then define

$$\underbrace{(\Phi^* \omega)_p}_{\in \Lambda^k T_p^* M,\ p \in M} (\underbrace{v_1, \cdots, v_k}_{\in T_p M}) := \underbrace{\omega_{f(p)}}_{\in \Lambda^k T^* N} (\underbrace{\Phi_* v_1, \cdots, \Phi_* v_k}_{\in T_{f(p)} N})$$

Prop. $\Phi^*(\alpha \wedge \beta) = (\Phi^* \alpha) \wedge (\Phi^* \beta)$.

Proof. local coordinate calculations.

Exterior derivative $d: \Lambda^k T^* M \to \Lambda^{k+1} T^* M$.

Definitions via local coordinates.

(a) For $f \in C^\infty(M)$ i.e. 0-form. Define $df = \frac{\partial f}{\partial u_i} du^i$ where $(u_1, \cdots, u_n)$ is the local coordinate.

(b) For $\omega \in T^* M$ i.e. 1-form. $\omega = \underbrace{\omega_i}_{\in C^\infty(M)} du^i$. define $d\omega := d\omega_i \wedge du^i = \frac{\partial \omega_i}{\partial u_j} du^j \wedge du^i$ using (a)

(c) For in general $\omega \in \Lambda^k T^* M$: $\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. $d\omega := d\omega_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

RMK. Definitions above are well-defined i.e. independent of local coordinates. For example, if $\{v_i\}_{i=1}^n$ is another local coordinate,

$$df = \frac{\partial f}{\partial u_i} du^i \overset{\text{chain rule}}{=} \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial u_i} \frac{\partial u^i}{\partial v_j} dv^j = \frac{\partial f}{\partial v_k} \Big(\underbrace{\frac{\partial u^i}{\partial v_j} \frac{\partial v^k}{\partial u_i}}_{\text{sum over } i,\ \delta_j^k} \Big) dv^j = \frac{\partial f}{\partial v_k} \delta_j^k dv^j = \frac{\partial f}{\partial v_j} dv^j$$

The others are similar. ∎

★ Really IMPORTANT: Theorem $\omega \in \Lambda^k T^* M$. $d(d\omega) = 0$. i.e. $d^2 = 0$.

Proof. $\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. $d\omega = \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

$$d(d\omega) = d\Big(\underbrace{\frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_j}}_{\in C^\infty(M)} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \Big) = d\Big(\frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_j} \Big) \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$= \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_j \partial x_r} dx^r \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = 0.$$

The last "=" is because $\frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_j \partial x_r} = \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_r \partial x_j}$ and $dx_r \wedge dx^j = -dx^j \wedge dx^r$ ∎

Def. Closed form if $d\omega = 0$, $\omega \in \ker(d_k)$

Exact form if $\exists\, \alpha \in \Lambda^{k-1} T^*M$ $\omega = d\alpha$. $\omega \in \operatorname{Im}(d_{k-1})$

By the last theorem, we have $\operatorname{Im}(d_{k-1}) \subseteq \ker(d_k)$

$0 \xrightarrow{d_0} T^*M \xrightarrow{d_1} \Lambda^2 T^*M \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} \Lambda^n T^*M \xrightarrow{d_n} 0$ is an exact sequence.

$H^k_{dR}(M) := \dfrac{\ker(d_k)}{\operatorname{Im}(d_{k-1})}$ is called the de Rham Cohomology group of M.

Prop. Simple properties of d.

1. $d(\omega + \eta) = d\omega + d\eta$, $\omega, \eta \in \Lambda^k T^*M$.
2. $d(f\omega) = df \wedge \omega + f\, d\omega$, $f \in C^\infty(M)$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ where $\alpha \in \Lambda^k T^*M$, $\beta \in \Lambda^r T^*M$.

"product rule".

4. $d_M(\underbrace{\Phi^*\omega}_{\in \Lambda^k T^*N}) = \Phi^*(d_N\omega)$ pullback commutes with exterior derivative

Proof of 4: For a 0-form $f \in C^\infty(M)$: $\Phi^* f = f \circ \Phi$. $d(\Phi^* f) = d(\underbrace{f\circ\Phi}_{\in C^\infty(M)}) = \dfrac{\partial(f\circ\Phi)}{\partial u_i} du^i = \left(\dfrac{\partial f}{\partial v_j}\circ\Phi\right)\dfrac{\partial v_j}{\partial u_i} du^i$ (chain rule.) while on the other hand,

$df = \dfrac{\partial f}{\partial v_i} dv^i$. $\Phi^*(df) = \dfrac{\partial f}{\partial v_i} \Phi^*(dv^i) = \left(\dfrac{\partial f}{\partial v_i}\circ\Phi\right)\Big|_p \dfrac{\partial v^i}{\partial u_j} du^j\Big|_p$.

The other steps:

Proposition 3.57. *Let $\Phi : M \to N$ be a smooth map between two smooth manifolds. For any $\omega \in \wedge^k T^*N$, we have:*

(3.20) $$\Phi^*(d\omega) = d(\Phi^*\omega).$$

To be precise, we say $\Phi^*(d_N\omega) = d_M(\Phi^*\omega)$, where $d_N : \wedge^k T^*N \to \wedge^{k+1} T^*N$ and $d_M : \wedge^k T^*M \to \wedge^{k+1} T^*M$ are the exterior derivatives on N and M respectively.

Proof. Let $\{u_j\}$ and $\{v_i\}$ be local coordinates of M and N respectively. By linearity, it suffices to prove (3.20) for the case $\omega = f\, dv^{i_1} \wedge \cdots \wedge dv^{i_k}$ where f is a locally defined scalar function. The proof follows from computing both LHS and RHS of (3.20):

$$d\omega = df \wedge dv^{i_1} \wedge \cdots \wedge dv^{i_k}$$
$$\Phi^*(d\omega) = \Phi^*(df) \wedge \Phi^*(dv^{i_1}) \wedge \cdots \wedge \Phi^*(dv^{i_k})$$
$$= d(\Phi^* f) \wedge d(\Phi^* v^{j_1}) \wedge \cdots \wedge d(\Phi^* v^{j_k}).$$

Here we have used Exercise 3.50. On the other hand, we have:

$$\Phi^*\omega = (\Phi^* f)\, \Phi^*(dv^{j_1}) \wedge \cdots \wedge \Phi^*(dv^{j_k})$$
$$= (\Phi^* f)\, d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k})$$
$$d(\Phi^*\omega) = d(\Phi^* f) \wedge d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k})$$
$$+ \Phi^* f\, d\left(d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k})\right)$$

Since $d^2 = 0$, each of $d(\Phi^* v^{i_q})$ is a closed 1-form. By Proposition 3.42 (product rule) and induction, we can conclude that:

$$d\left(d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k})\right) = 0$$

and so $d(\Phi^*\omega) = d(\Phi^* f) \wedge d(\Phi^* v^{i_1}) \wedge \cdots \wedge d(\Phi^* v^{i_k})$ as desired. $\square$

Lie derivative.

We first introduce the "dynamical" definition. For referencing purpose, here we recall some basic theorems of ODEs in $\mathbb{R}^n$.

From ODEs by Frederick Fong. 2020.

Theorem 2.6 (Picard-Lindelöf's Existence Theorem). *Let Ω be an open domain in $\mathbb{R}^d$, $\mathbf{x}_0$ be a point in Ω, and $I = [-T, T]$ be a closed and bounded time interval. Suppose $\mathbf{F}(\mathbf{x}, t) : \Omega \times I \to \mathbb{R}^d$ is a vector field which is locally Lipschitz continuous on $\Omega \times I$ (see Definition 2.7), then the initial-value problem*

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

has a solution $\mathbf{x}(t)$ defined on an interval $[-\varepsilon, \varepsilon] \subset I$ for some $\varepsilon > 0$.

Similarly, for an autonomous system with a vector field $\mathbf{G} : \Omega \to \mathbb{R}^d$ which is locally Lipschitz continuous on Ω, the IVP

$$\mathbf{x}' = \mathbf{G}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0$$

has a solution defined on an interval $[-\varepsilon', \varepsilon']$ for some $\varepsilon' > 0$.

Remark 2.17. By Theorem 2.3, any C^1 vector field must be locally Lipschitz continuous. Therefore, Theorem 2.6 applies to all C^1 vector fields. Many examples we have seen so far are C^1 on their domain. □

Theorem 2.9 (Continuous Dependence Inequality for Nonlinear Systems). *Let $\Omega \subset \mathbb{R}^d$ be an open domain and I be a time interval. Suppose $\mathbf{F}(\mathbf{x}, t) : \Omega \times I \to \mathbb{R}^d$ is a vector field which is Lipschitz continuous on $\Omega \times I$ with a Lipschitz constant L. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, are both solutions to the system $\mathbf{x}' = \mathbf{F}(\mathbf{x}, t)$, and $\mathbf{x}_1(t), \mathbf{x}_2(t) \in \Omega$ for t in some interval $I' \subset I$, then we have:*

$$|\mathbf{x}_1(t) - \mathbf{x}_2(t)| \leq |\mathbf{x}_1(t_0) - \mathbf{x}_2(t_0)|\, e^{L|t-t_0|} \tag{2.14}$$

for any $t_0, t \in I'$.

Remark 2.23. In simpler terms, the inequality (2.14) holds as long as both solutions stay inside Ω. □

Corollary 2.2 (Uniqueness Theorem: Lipschitz). *Suppose $\mathbf{F}(\mathbf{x}, t) : \Omega \times I \to \mathbb{R}^d$ be a vector field which is Lipschitz continuous on $\Omega \times I$ and $\mathbf{x}_0$ is a point in Ω. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, defined on $t \in I' \subset I$ such that $\mathbf{x}_1(t), \mathbf{x}_2(t) \in \Omega$ for $t \in I'$, are both solutions to the IVP:*

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

then we have $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ for all $t \in I'$.

Remark 2.26. In simpler terms, the corollary asserts that the solution to an IVP is unique as long as the solution lies in Ω. □

Corollary 2.3 (Uniqueness Theorem: Locally Lipschitz). *Suppose $\mathbf{F}(\mathbf{x}, t) : \Omega \times I \to \mathbb{R}^d$ be a vector field which is locally Lipschitz continuous on $\Omega \times I$. Let $\mathbf{x}_0$ be a point in Ω. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, defined on $t \in I' \subset I$ such that $\mathbf{x}_1(t), \mathbf{x}_2(t) \in \Omega$ for $t \in I'$, are both solutions to the IVP:*

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

then we have $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ for all $t \in I'$.

Definition. Local flow.

By those theorems, we can construct a unique local flow of a smooth vector field X to be: $\Phi_t(p) := \varphi(p,t)$

where $\varphi: M \times (-\varepsilon, \varepsilon) \to M$ smooth. $\varphi(p,0) = p \in M$. $\frac{d}{dt}\varphi(p,t) = X(\varphi(x_0,t))$. $\forall t \in (-\varepsilon, \varepsilon)$.

Note it follows from the uniqueness theorem that $\Phi_t \circ \Phi_s = \Phi_{t+s}$ $\forall t, s \in \mathbb{R}$ $t+s \in (-\varepsilon, \varepsilon)$. (just show both sides satisfy the equation)

$\Rightarrow \Phi_t^{-1} = \Phi_{-t}$. $\forall$ fixed $p \in M$. $\Phi_{(\cdot)}(\ ) : M \to M$ is a differomorphism, so called the One parameter group of differomorphism.

Definition. $\mathcal{L}_X f$. Given a smooth function f on M, we define The Lie derivative of f along X at $p \in M$,

$$\mathcal{L}_X f\big|_p := \lim_{t \to 0} \frac{f(\Phi_t(p)) - f(p)}{t} \qquad \text{So } \mathcal{L}_X : C^\infty(M) \longmapsto C^\infty(M).$$

Theorem. $\mathcal{L}_X f = X(f)$. The proof is done via local coordinates:

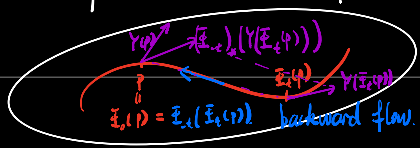
In terms of local coordinates: suppose $F(u_1, \cdots, u_n)$ is a local chart around $p = F(0, \cdots, 0)$. $X = X^i \frac{\partial}{\partial u_i}$. WLOG we could assume $\Phi_t(p) = \Phi_t \circ F(u_1(t), \cdots, u_n(t))$ is also in the chart

$$\mathcal{L}_X f|_p = \lim_{t\to 0} \frac{f(\Phi_t(p)) - f(p)}{t} = \lim_{t\to 0} \frac{f\circ F(u_1(t), \cdots, u_n(t)) - f\circ F(0, \cdots, 0)}{t}$$

$$\overset{\text{chain rule}}{=} \frac{\partial (f\circ F)}{\partial u_i} \frac{du_i}{dt}\Big|_p = X^i \frac{\partial (f\circ F)}{\partial u_i} = X^i \frac{\partial}{\partial u_i}(f) = X(f).$$

So we conclude. $\mathcal{L}_X(f) = X(f) = X^i \frac{\partial f}{\partial u_i}$ is just the directional derivative of f along X.

And then we consider compare two vectors $X(p)$ and $X(\Phi_t(p))$ by transform both vectors to the same vector space:



push $Y(\Phi_t(p))$ back to T_pM.

Given a vector field Y over M. $Y(p) \in T_pM$ $Y(\Phi_t(p)) \in T_{\Phi_t(p)}M$ $\Rightarrow (\Phi_{-t})_*(Y(\Phi_t(p))) \in T_pM$. here we use push-foward of Φ_{-t} to move tangent vectors at $\Phi_t(p)$ to p. Then we define.

The Lie derivative of a vector field Y along X at $p \in M$: $\mathcal{L}_X Y|_p := \lim_{t\to 0} \frac{(\Phi_{-t})_*(Y(\Phi_t(p))) - Y(p)}{t}$.

Before we do local calculations, we define the Lie bracket $[X, Y] = XY - YX$.

In terms of local coordinates. $[X, Y] = XY - YX = X^i \frac{\partial}{\partial u_i}\left(Y^j \frac{\partial}{\partial u_j}\right) - Y^j \frac{\partial}{\partial u_j}\left(X^i \frac{\partial}{\partial u_i}\right)$

$$= X^i\left(\frac{\partial Y^j}{\partial u_i}\frac{\partial}{\partial u_j} + \cancel{Y^j \frac{\partial^2}{\partial u_i \partial u_j}}\right) - Y^j\left(\frac{\partial X^i}{\partial u_j}\frac{\partial}{\partial u_i} + \cancel{\frac{\partial^2}{\partial u_i \partial u_j} \cdot X^i}\right)$$

$$= X^i \frac{\partial Y^j}{\partial u_i}\frac{\partial}{\partial u_j} - Y^j \frac{\partial X^i}{\partial u_j}\frac{\partial}{\partial u_i} \overset{j \to i}{=\!=} \left(X^j \frac{\partial Y^i}{\partial u_j} - Y^j \frac{\partial X^i}{\partial u_j}\right)\frac{\partial}{\partial u_i}$$

$$\Rightarrow [X, Y] = \left(X^j \frac{\partial Y^i}{\partial u_j} - Y^j \frac{\partial X^i}{\partial u_j}\right)\frac{\partial}{\partial u_i}$$

Theorem. We could similarly verify the following properties:

(a) antisymmetric: $[X, Y] = -[Y, X]$. (b) $\mathbb{R}$-Bilinearity

(c) Jacobian identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

(d) Product rule: $[fX, gY] = fg[X,Y] + f\cdot X(g)Y - g\cdot Y(f)X$

Theorem. $\mathcal{L}_X(Y) = [X, Y]$.

proof. there is a proof using local coordinates but it's quite complicated. Here we adapt a proof from P. Petersen without using local coordinates.

By the def of $\mathcal{L}_X Y$, we have $(\Phi_{-t})_*(Y(\Phi_t(p))) - Y(p) = t\mathcal{L}_X Y + o(t)$. $(*)$

Use $(\Phi_t)_*$ to act on both sides:

$\Leftrightarrow \quad Y(\Phi_t(p)) - (\Phi_t)_*(Y(p)) = t(\Phi_t)_*(\mathcal{L}_X Y) + o(t). \quad (**)$

Pick $f \in C^\infty$. Consider the directional derivative of f at $\Phi_t(p)$ along RHS of $(**)$:

$$\Big[Y(\Phi_t(p)) - (\Phi_t)_*(Y(p))\Big](f) = Y(\Phi_t(p))(f) - (\Phi_t)_*(Y(p))(f)$$

$$= Y(f)\big|_{\Phi_t(p)} - Y(f|_{\Phi_t(p)}).$$

$$= \cancel{Y(f)} + t\,XY(f) + o(t)$$

$$- Y(\cancel{f} + t\,X(f) + o(t))$$

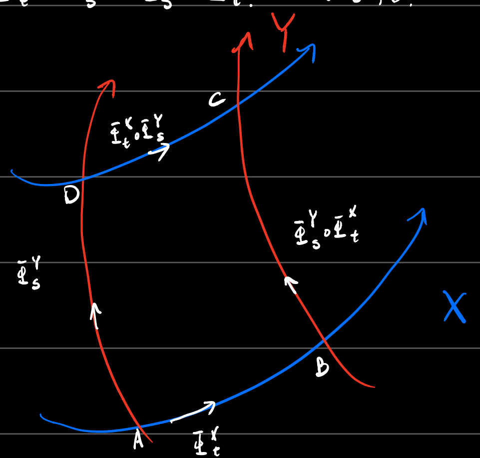
all evaluated at p.

$$= t\,(XY - YX)(f) + o(t) \qquad (***)$$

$\Rightarrow \mathcal{L}_X Y = [X, Y]$. by comparing $(*)$ and $(***)$ ∎

Corollary. If $X, Y \in \mathcal{X}(M)$ the space of smooth vector field. $[X, Y] = 0$

then $\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$. $\forall s, t$.



Definition. Lie derivative of 1-form. Unlike vector field case we use backward push-forward to move vectors, now we could directly apply pullback of 1-forms to drag it to p:

Let α be a 1-form. $\mathcal{L}_X \alpha|_p := \lim_{t\to 0} \dfrac{(\Phi_t)^*(\alpha(\Phi_t(p))) - \alpha(p)}{t}$ where $\Phi_t(p)$ is the local flow of X.

In terms of local coordinates, if $\alpha = \alpha_i du^i$, then $\Phi_t^*(\alpha|_{\Phi_t(p)}) = \alpha_i|_{\Phi_t(p)} \Phi_t^*(du^i) = \alpha_i|_{\Phi_t(p)} \dfrac{\partial u^i}{\partial u_j} du^j$ and hence

$$\mathcal{L}_X \alpha = \frac{d}{dt}\Big|_{t=0} \Big(\alpha_i|_{\Phi_t(p)} \frac{\partial u^i}{\partial u_j} du^j\Big) = \Big(\frac{\partial \alpha_i}{\partial t}\frac{\partial u^i}{\partial u_j} + \alpha_i \frac{\partial^2 u_i}{\partial u_j \partial t}\Big)\Big|_{t=0} du^j$$

$$= \Big(\frac{\partial \alpha_i}{\partial u_k}\boxed{\frac{\partial u_k}{\partial t}}\underbrace{\frac{\partial u^i}{\partial u_j}}_{\text{at } t=0,\ \frac{\partial u^i}{\partial u_j} = \delta^i_j} + \alpha_i \cdot \overset{X^i}{\frac{\partial^2 u_i}{\partial u_j \partial t}}\Big)\Big|_{t=0} du^j$$

$$= \Big(\frac{\partial \alpha_j}{\partial u_k} X^k + \alpha_i \frac{\partial X^i}{\partial u_j}\Big) du^j$$

$\Rightarrow \mathcal{L}_X \alpha = \left(X^j \frac{\partial \alpha_i}{\partial u_j} + \alpha_j \frac{\partial X^j}{\partial u_i} \right) du^i$ ∎

Definition. Lie derivative of $(k,0)$-tensor field.

Now we generalize Lie derivative of 1-forms (1,0-tensors) to $(k,0)$ tensors: Suppose $T = T_{i_1 \cdots i_k} du^{i_1} \cdots du^{i_k}$ is a $(k,0)$ tensor on M.

$$\mathcal{L}_X T := \mathcal{L}_X (T_{i_1 \cdots i_k} du^{i_1} \cdots du^{i_k}) = \underbrace{\mathcal{L}_X (T_{i_1 \cdots i_k})}_{\text{Lie derivative of a smooth function } = X(T_{i_1 \cdots i_k})} du^{i_1} \cdots du^{i_k} + \sum_{j=1}^{k} T_{i_1 \cdots i_k} du^{i_1} \otimes \cdots \otimes \mathcal{L}_X (du^{i_j}) \otimes \cdots \otimes du^{i_k}$$

As we already define $\mathcal{L}_X (du^{i_j})$, our new def. makes sense. One may also compute the local coordinates.

We now give the axiomatic definition.

Axiom 1. $\mathcal{L}_X f = X(f) \quad \forall f \in C^\infty(M)$

2. $\mathcal{L}_X (T \otimes S) = \mathcal{L}_X(T) \otimes S + T \otimes \mathcal{L}_X(S)$. product rule

3. $\mathcal{L}_X (\underset{(n,0)\text{ tensor}}{T} (Y_1, \cdots, Y_n)) = (\mathcal{L}_X T)(Y_1, \cdots, Y_n) + T(\mathcal{L}_X(Y_1), \cdots, Y_n) + \cdots + T(Y_1, \cdots, \mathcal{L}_X(Y_n))$.

4. $\mathcal{L}_X$ commutes with exterior derivative d on differential forms.

RMK. $\mathcal{L}_X$ does not change the tensor type.

We have something to say about 4. above. There is a nice formula for $\mathcal{L}_X$ acting on differential forms.

Cartan's magic formula. $\mathcal{L}_X \alpha = i_X (d\alpha) + d(i_X \alpha)$. $\forall \alpha \in \Lambda^k T^* M$.

Definition. Given vector field X. $\omega \in \Lambda^k T^* M$. define the interior product $i_X : \Lambda T^* M \longrightarrow \Lambda^{k-1} T^* M$ given by

$$\underset{\text{vector field}}{i_X} \underset{k\text{-form}}{\omega} (Y_1, \cdots, Y_{k-1}) = \underset{k\text{-form}}{\omega}(X, Y_1, \cdots, Y_{k-1}).$$

$(k-1)$-form.

Properties of i_X:

1) linearity 2) $i_X i_Y \omega = - i_Y i_X \omega$. by antisymmetry of diff. forms. 3) $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$. 4) $i_X(\underset{p\text{-form}}{\alpha} \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^p \alpha \wedge i_X(\beta)$ (product rule)

Proof of sketch of Cartan's magic formula.

Induction. for 1-form $\alpha = \alpha_i du^i$: $d\alpha = d\alpha_i \wedge du^i = \frac{\partial \alpha_i}{\partial u_j} du^j \wedge du^i$ $\quad \mathcal{L}_X \alpha = \left(X^j \frac{\partial \alpha_i}{\partial u_j} + \alpha_j \frac{\partial X^j}{\partial u_i} \right) du^i$ as we've done before.

$\Rightarrow i_X(d\alpha) = \frac{\partial \alpha_i}{\partial u_j} i_X(du^j \wedge du^i) = \frac{\partial \alpha_i}{\partial u_j} \left(\cancel{-X^i du^j} + X^j du^i \right)$

$i_X(du^i du^j)(Y) = du^i(X) du^j(Y) = X^i Y^j \Rightarrow i_X(du^i du^j) = X^i du^j$

$\Rightarrow i_X(du^j \wedge du^i) = i_X(du^j du^i) - i_X(du^i du^j) = -X^i du^j + X^j du^i$

$i_X \alpha = \alpha(X) = \alpha_i X^i$. $\quad d(i_X \alpha) = \frac{\partial(\alpha_i X^i)}{\partial u_j} du^j = \left(\cancel{X^i \frac{\partial \alpha_i}{\partial u_j}} + \alpha_i \frac{\partial X^i}{\partial u_j} \right) du^j$

$\Rightarrow \quad i_X(d\alpha) + d(i_X\alpha) = \frac{\partial \alpha_i}{\partial u_j} X^j du^i + \alpha_i \frac{\partial X^i}{\partial u_j} du^j \xlongequal[i \leftrightarrow j]{\text{relabelling}} \mathcal{L}_X \alpha$

For general case, assume Cartan's formula holds for k-form. WLOG Let $\omega = \omega\, dx^1 \wedge \cdots \wedge dx^k \in \Lambda^k T^*M$

To be continued....

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