Reference: P. Peterson's Note. Fredrick Fong's Note. ICTP's video on DG. Do Carmo 1976

Ref. Elementry Diff. Geo. Andrew Pressley.

Def. local chart and regular surface.

$M \subseteq \mathbb{R}^3$ is called a regular surface if $\forall p \in M$. $\exists\, O \ni p$ open subset of $\mathbb{R}^3$, a map $\varphi: U \longrightarrow O \cap M$. s.t. ($U \subseteq \mathbb{R}^2$)

1) φ is smooth regarding as a map $U \to \mathbb{R}^3$ 2) φ is bijective and φ^{-1} is continuous. 3) $\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \neq 0 \quad \forall (u,v) \in U$.

Remark. The third requirement is equivalent to: $\left(\frac{\partial \varphi}{\partial u}\ \frac{\partial \varphi}{\partial v}\right)_{3\times 2}$ has rank 2. $\Longleftrightarrow$ $d\varphi_p$ is injective from $\mathbb{R}^2 \longrightarrow T_pS$. (Defined later)

Level set Theorem.

Theorem 1.6. *Let $g(x,y,z): \mathbb{R}^3 \to \mathbb{R}$ be a smooth function of three variables. Consider a non-empty level set $g^{-1}(c)$ where c is a constant. If $\nabla g(x_0,y_0,z_0) \neq 0$ at all points $(x_0,y_0,z_0) \in g^{-1}(c)$, then the level set $g^{-1}(c)$ is a regular surface.*

Proof: a simple application of Inverse function Thm.

Inverse Func. Thm.

THEOREM 4.1.5. *Let $\mathrm{q}(u,v): U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface. For every $(u_0,v_0) \in U$ there exists a neighborhood $(u_0,v_0) \in V \subset U$ such that the smaller parametrized surface $\mathrm{q}(u,v): V \to \mathbb{R}^3$ can be represented as a Monge patch.*

Def. Let S be a regular surface $\subseteq \mathbb{R}^3$. O is open set of $\mathbb{R}^3$.

A) $F: S \to \mathbb{R}$ differentiable if $\forall$ local paramtrization $X: U \to S$, $F \circ X: U \to \mathbb{R}$ is differentiable. ($U \subseteq \mathbb{R}^2$)

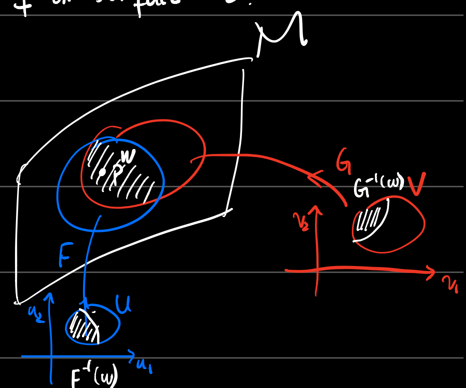
B) $f: O \to S$ is diff. if $f: O \to \mathbb{R}^3$ is diff. in the usual sense.

C) $f: M \to N$ is diff. if $f: M \to \mathbb{R}^3$ is diff. in the sense of A.

i.e. $f \circ X: U \to \mathbb{R}^3$ is diff.

Note: If $f: O \to \mathbb{R}^3$ smooth, so is $S \subseteq O$. $f|_S: S \to \mathbb{R}^3$ restriction of f on surface S.

Transition maps between local charts.



Proposition 1.11. *Let $M \subset \mathbb{R}^3$ be a regular surface, and $F_\alpha(u_1,u_2): \mathcal{U}_\alpha \to M$ and $F_\beta(v_1,v_2): \mathcal{U}_\beta \to M$ be two smooth local parametrizations of M with overlapping images, i.e. $\mathcal{W} := F_\alpha(\mathcal{U}_\alpha) \cap F_\beta(\mathcal{U}_\beta) \neq \emptyset$. Then, the transition maps defined below are also smooth maps:*

$$(F_\beta^{-1} \circ F_\alpha): F_\alpha^{-1}(\mathcal{W}) \to F_\beta^{-1}(\mathcal{W})$$
$$(F_\alpha^{-1} \circ F_\beta): F_\beta^{-1}(\mathcal{W}) \to F_\alpha^{-1}(\mathcal{W})$$

Proof. We just prove the first one: $G^{-1} \circ F: F^{-1}(W) \to G^{-1}(W)$ is smooth.

$(v_1, v_2) = G^{-1} \circ F(u_1, u_2)$ By assumption, $\frac{\partial F}{\partial u_1} \times \frac{\partial F}{\partial u_2} \neq 0 \Leftrightarrow \left(\det\left(\frac{\partial y}{\partial u}\ \frac{\partial z}{\partial u}\right), \det\left(\frac{\partial x}{\partial u}\ \frac{\partial z}{\partial u}\right), \det\left(\frac{\partial y}{\partial u}\ \frac{\partial x}{\partial u}\right)\right) \neq 0$.

Suppose $\frac{\partial(x,y)}{\partial(u_1,u_2)} \neq 0$. by $\hat{k}$-component of normal vector $\Longleftrightarrow \frac{\partial(x,y)}{\partial(v_1,v_2)} \neq 0$.

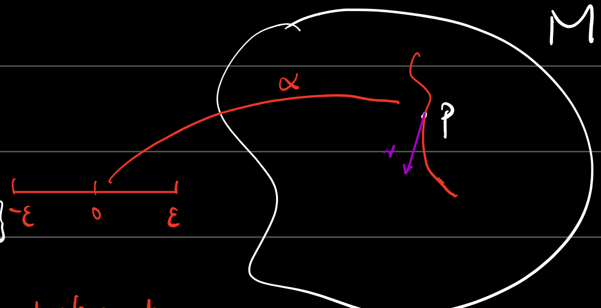
$\Rightarrow$ $\pi_z(x,y,z) := (x,y)$ then $\pi_z \circ G(v_1,v_2) = (x(u_1,u_2), y(u_1,u_2))$ has Jacobian $\left[\frac{\partial x}{\partial v_1}\ \frac{\partial y}{\partial v_2}\right]$ non zero. at p.

$\Rightarrow$ $\pi \circ G$ locally invertible. with smooth inverse.

$G^{-1} \circ F = (\pi \circ G)^{-1} \circ (\pi \circ F)$ is then smooth. ∎

As a Corollary, the "$\forall$ parametrization" in the previous definition can be replaced by "$\exists$".

Tangent Space.



Def. $T_pM := \{ v \in \mathbb{R}^3 \mid \exists\, \alpha : (-\varepsilon, \varepsilon) \longrightarrow M.\ \alpha(0) = p \quad \alpha'(0) = v \}$

$= \mathrm{Span}\left\{ \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v} \;\middle|\; \varphi : U \to M \text{ is a local chart around } p \right\}$

$=$ image of $d\varphi_{\varphi^{-1}(p)}(\mathbb{R}^2)$. **Warning.** T_pS is NOT tangent plane at p.

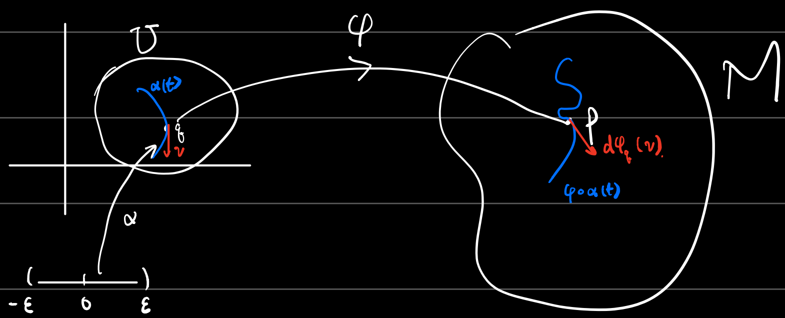
they are parallel and T_pS passes through the origin so that it's a subspace.

Example $S = \{ 2z = x^2 + y^2 \} = \Gamma_f$ where $f(x,y) = \frac{1}{2}(x^2+y^2)$ is a regular surface.

A global parametrization is $\varphi(u,v) = (u,\ v,\ \frac{1}{2}(u^2+v^2))$ $T_pS = \mathrm{span}\left\{ \frac{\partial \varphi}{\partial u}\Big|_p, \frac{\partial \varphi}{\partial v}\Big|_p \right\} = \mathrm{span}\{ (1,0,u_0), (0,1,v_0) \}$ if $p = \varphi(u_0, v_0)$.

Differential of a map at a point.

(A) $M \subseteq \mathbb{R}^3$ regular surface. $p \in M$. $\varphi : U \longmapsto \varphi(U) \ni p$ is a local chart around p. $\varphi(q) = p$. $d\varphi_q : \mathbb{R}^2 \to \mathbb{R}^3$ defined as:



Given $v \in \mathbb{R}^2$, $\exists\, \alpha : (-\varepsilon, \varepsilon) \to U$. s.t.

$\alpha(0) = q$ and $\alpha'(0) = v$

$$d\varphi_q(v) = \frac{d}{dt}\Big|_{t=0} (\varphi \circ \alpha)(t).$$

* **Thm.** This def. is independent of the curve α. (well-defined).

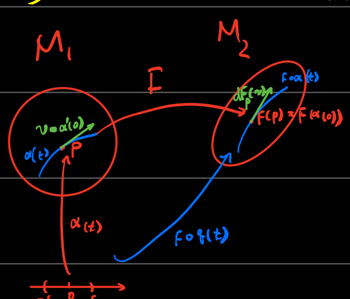
Proof. Let's compute $d\varphi_q$ directly:

$$d\varphi_q(v) = \frac{d}{dt}\Big|_{t=0} (\varphi \circ \alpha(t)) \overset{\text{Chain rule}}{=} \begin{pmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \end{pmatrix}_{\alpha(0)} \cdot \begin{pmatrix} \alpha'(0) \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \end{pmatrix}_{\underset{\alpha(0)=q}{q}} \begin{pmatrix} \overset{\alpha'(0)=v}{v} \end{pmatrix}$$

So we only need information q and v.

Further, it show that $d\varphi_q : \mathbb{R}^2 \to \mathbb{R}^3 = \begin{pmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \end{pmatrix}_q$ is linear and injective (by def. of regular surface) ∎

(B) $F : M_1 \to M_2$ smooth. $dF_p : T_pM_1 \longrightarrow T_{F(p)}M_2$ is defined by:



$$dF_p(v) = \frac{d}{dt}\Big|_{t=0} \left(F \circ \alpha(t) \right)$$

$dF_p(v) = \begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \end{pmatrix}_p \begin{pmatrix} v \end{pmatrix}$

Def. Regular value. $a \in \mathbb{R}$ is a regular value of $f : M \to \mathbb{R}$ if $df_p \neq 0 \quad \forall p \in f^{-1}(a)$

Critical point of M. $p \in M$ is critical if $df_p \equiv 0$. is the zero linear map: $df_p(v) = 0 \quad \forall v \in T_pM$.

*Thm. (Chain rule) $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ Then $d(f\circ g)_p = dg_q \circ df_p$ where $q = f(p)$
$p \to q \to r$.

Proof. Pick $p \in M_1$, $v \in T_pM_1$, then $\exists\, \alpha(t): (-\varepsilon,\varepsilon) \to M_1$ s.t. $\alpha(0)=p$ $\alpha'(0)=v$. then

- $\beta(t) = f\circ\alpha(t): (-\varepsilon,\varepsilon) \to M_2$ is a curve on M_2 with $\beta'(0) = \frac{d}{dt}\Big|_{t=0}(f\circ\alpha(t))$ = tangent vector at q. $g\circ f\circ\alpha(t)$ is a curve on M_3.

- $d(g\circ f)_p(v) = \frac{d}{dt}(g\circ f\circ\alpha(t))$

$dg_q\circ df_p(v) = dg_q\Big(\underbrace{\frac{d}{dt}\Big|_{t=0}(f\circ\alpha(t))}_{\substack{\text{vector in } T_qM_2 \\ \beta'(0)}}\Big) = \frac{d}{ds}\Big|_{s=0}\big(g\circ\beta(s)\big) = \frac{d}{ds}\Big|_{s=0}(g\circ f\circ\alpha(s)) = d(g\circ f)_p(v)$.

Examples. 1) Let A be real symmetric 4×4 matrix. $S = \{v\in\mathbb{R}^3 \mid (1\ v^T)A\binom{1}{v} = 0\}$.

Define $f(v) = (1\ v^T)A\binom{1}{v}$, then $S = f^{-1}(0)$. S is a regular surface iff 0 is a regular value of $f \Leftrightarrow df_p \neq 0$ $\forall p\in S$.

So Let's compute $df_p(\underset{\in T_pS}{v})$: Take a curve $\alpha: (-\varepsilon,\varepsilon)\to S$. $\alpha(0)=p$ $\alpha'(0)=v$. Then $f\circ\alpha(t) = (1\ \alpha(t)^T)A\binom{1}{\alpha(t)}$.

By definition, $df_p(v) = \frac{d}{dt}\Big|_{t=0}(f\circ\alpha(t)) \overset{\text{product derivative rule}}{=} (0\ \alpha'(0)^T)A\binom{1}{\alpha(0)} + (1\ \alpha(0)^T)A\binom{0}{\alpha'(0)} \overset{A \text{ symmetric}}{=} 2(1\ \underset{\alpha(0)}{p^T})A\binom{0}{\underset{\alpha'(0)}{v}}$

$df_p \equiv 0 \Leftrightarrow df_p(v)=0\ \forall v. \Leftrightarrow (1\ p^T)A = (\lambda, \vec{0})$ for some $\lambda\in\mathbb{R}$. But $p\in f^{-1}(0)$. So $(1\ p^T)A\binom{1}{p} = 0 \Rightarrow (\lambda,\vec{0})\binom{1}{p} = 0 \Rightarrow \lambda = 0$

Hence, 0 is NOT a regular value $(1\ X^T)A = \vec{0}$ has solution.

2) $f: O\to\mathbb{R}$ differentiable. $a\in\mathbb{R}$ is a regular value. $S = f^{-1}(a)$ a regular surface. Then: $T_pS = \ker(df_p: \mathbb{R}^3\to\mathbb{R})$

Proof. "$\subseteq$": Pick $v\in T_pS$. $\exists\,\alpha: (-\varepsilon,\varepsilon)\to S$. $\alpha(0)=p$ $\alpha'(0)=v$

$df_p(v) = \frac{d}{dt}\Big|_{t=0}(f\circ\underset{\substack{\alpha(t)\in S \\ f(\alpha(t))=a\ \forall t}}{\alpha(t)}) = \frac{d}{dt}\Big|_{t=0} a = 0. \Rightarrow v\in\ker(df_p)$.

"$\supseteq$": $T_pS \subseteq \underbrace{\ker(df_p)}$. both are 2-dimension spaces so they have to be equal.
2D since a regular value, and $df_p: \mathbb{R}^3\to\mathbb{R}$

3) $S^2_a(r) = \{p\in\mathbb{R}^3 \mid |p-a|^2 = r^2\}$ Define $f: \mathbb{R}^3\to\mathbb{R}$ by $f(p) = |p-a|^2$. Then $S_a(r) = f^{-1}(r^2)$.

By 2),
$T_pS^2_a(r) = \ker(df_p) = \{v\in\mathbb{R}^3 \mid df_p(v)=0\} = \{v\in\mathbb{R}^3: v\perp\vec{a}\}$ agrees with our usual intuition.

Compute
$df_p(v) \overset{\text{def.}}{=} \frac{d}{dt}\Big|_{t=0}(f\circ\alpha(t)) = \frac{d}{dt}\Big|_{t=0}|\alpha(t)-a|^2 = \frac{d}{dt}\Big|_{t=0}\langle\alpha(t)-a, \alpha(t)-a\rangle = 2\langle\alpha'(0), a\rangle = 2\langle v, a\rangle$

4) Let A be a symmetric 3×3 real matrix. Using the notion of diff. map to show it's orthogonally diagonalizable with 3 eigenvalues

Define $f: S^2\to\mathbb{R}$ by $f(p) = \langle Ap, p\rangle$. Critical point of $f = \{p\in S^2 \mid df_p \equiv 0\}$. Compute df_p:

$df_p(v) = \frac{d}{dt}\Big|_{t=0}(f\circ\alpha(t)) = \frac{d}{dt}\Big|_{t=0}\langle A\alpha(t), \alpha(t)\rangle = 2\langle A\alpha(0), \alpha'(0)\rangle = 2\langle Ap, v\rangle$

p is critical point

$\Updownarrow$

$df_p \equiv 0 \Leftrightarrow df_p(v) = 0 \quad \forall v \in T_pS^2. \Rightarrow Ap \perp T_pS^2 \Rightarrow Ap = \lambda p$ for some $\lambda \in \mathbb{R}$. Since $p \perp T_pS^2 \Leftrightarrow p$ is eigenvector of A with eigenvalue λ. But $f(p) = \langle Ap, p \rangle = \lambda$. So p is critical $\Leftrightarrow f(p)\,p = Ap$

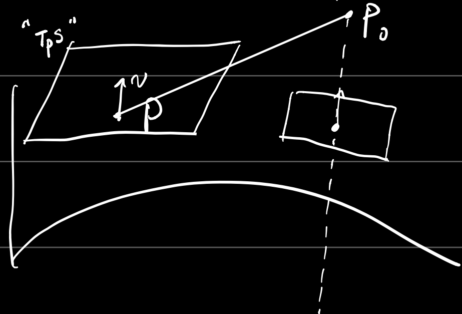
- If f is const: $A = \mu\,\mathrm{id}$ trivial
- If f is not const. S^2 is compact so f achieves it's min and max, which are exactly the critical points of f:

$\{p_1, p_2\}$ is unit vectors and $p_1 \perp p_2$: indeed, $[f(p_1) - f(p_2)] \langle p_1, p_2 \rangle = \langle f(p_1)p_1, p_2 \rangle - \langle p_1, f(p_2)p_2 \rangle = \langle Ap_1, p_2 \rangle - \langle p_1, Ap_2 \rangle = 0$ since A is symmetric

The only one missing must be $\vec{n} = p_1 \times p_2$.

5) Fix $p_0 \in \mathbb{R}^3$. $f(p) = |p_0 - p|^2$. $f: S^2 \to \mathbb{R}$. then $df_p(v) = \frac{d}{dt}\Big|_{t=0} (f \circ \alpha(t)) = 2 \langle \alpha(0) - p_0, \alpha'(0) \rangle = 2 \langle p - p_0, v \rangle$

hence p is critical $\Leftrightarrow p_0 - p \perp v \quad \forall v \in T_pS^2 \Leftrightarrow$ the normal line at p passes p_0.



First Fundamental Form (Riemannian Metric)

Definition. $I_p(\cdot)$ is a quadratic form on T_pM defined as $I_p(v) = \langle v, v \rangle = |v|^2$. where $\langle , \rangle$ is the usual dot product in $\mathbb{R}^3$.

local coordinate expression: If $F(u,v)$ is a local chart around $p \in M$. then $\forall v \in T_pM$, $\exists\ \alpha: (-\varepsilon, \varepsilon) \to M$ $\alpha(0) = p$. $\alpha'(0) = v$.

Suppose $F(u_0, v_0) = p$. Write $\alpha(t) = F(u(t), v(t))$. $v = v^u \frac{\partial F}{\partial u} + v^v \frac{\partial F}{\partial v}$

$I_p(v) = \langle v, v \rangle = \langle \alpha'(0), \alpha'(0) \rangle = \langle \frac{\partial F}{\partial u} u'(0) + \frac{\partial F}{\partial v} v'(0), \frac{\partial F}{\partial u} u'(0) + \frac{\partial F}{\partial v} v'(0) \rangle = (v^u)^2 \langle F_u, F_u \rangle + 2 v^u v^v \langle F_u, F_v \rangle + (v^v)^2 \langle F_v, F_v \rangle$

In general. consider the corresponding bilinear form $I: T_pM \times T_pM \longrightarrow \mathbb{R}$

$$[I] = \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial u} & \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v} \\ \frac{\partial F}{\partial v} \cdot \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \cdot \frac{\partial F}{\partial v} \end{bmatrix}.$$

$I(X, Y) = X^T [I] Y$ Now use local coordinate, $\begin{cases} X = X^u \frac{\partial F}{\partial u} + X^v \frac{\partial F}{\partial v} = \begin{bmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \end{bmatrix} \begin{bmatrix} X^u \\ X^v \end{bmatrix} \\ Y = Y^u F_u + Y^v F_v = \begin{bmatrix} F_u & F_v \end{bmatrix} \begin{bmatrix} Y^u \\ Y^v \end{bmatrix}. \end{cases}$

$$I(X, Y) = \begin{bmatrix} X^u & X^v \end{bmatrix} \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} Y^u \\ Y^v \end{bmatrix}$$

RMK. Although $[I]$ dependes on parametrization F, $I(X, Y)$ does not since T_pM does not depends on F.

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Example 4. We shall compute the first fundamental form of a sphere at a point of the coordinate neighborhood given by the parametrization (cf. Example 1, Sec. 2-2)

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

$$\frac{\partial X}{\partial \theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \qquad \frac{\partial X}{\partial \varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$[I] = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}.$$

So $v \in T_pS^2$, if $v = a X_\theta + b X_\varphi$. $I(v) = |v|^2 = a^2 + b^2\sin^2\theta$.

Theorem. (Area form)

Another familiar geometric quantity which is also related to g is the area of a surface. For simplicity, we focus on dimension 2 first. Suppose a regular surface Σ can be almost everywhere parametrized by $F(u, v)$ with $(u, v) \in D \subset \mathbb{R}^2$ where D is a bounded domain, the area of this surface is given by:

$$A(M) = \iint_D \left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right| dudv$$

It is also possible to express $|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}|$ in terms of the first fundamental form g. Let θ be the angle between the two vectors $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, then from elementary vector geometry, we have:

$$\begin{aligned}
\left|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right|^2 &= \left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 \sin^2\theta \\
&= \left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 - \left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 \cos^2\theta \\
&= \left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 - \left(\frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v}\right)^2 \\
&= \left(\frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial u}\right)\left(\frac{\partial F}{\partial v} \cdot \frac{\partial F}{\partial v}\right) - \left(\frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v}\right)^2 \\
&= g_{11}g_{22} - (g_{12})^2 \\
&= \det[g].
\end{aligned}$$

Therefore,

(7.2) $$A(M) = \iint_D \sqrt{\det[g]}\, dudv$$

Thm. arc-length on surface.

Consider a curve γ on a regular hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ parametrized by $F(u_i)$. Suppose the curve can be parametrized by $\gamma(t)$, $a < t < b$, then from calculus we know the arc-length of the curve is given by:

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

In fact one can express this above in terms of g. The argument is as follows:

Suppose $\gamma(t)$ has local coordinates coordinates $(\gamma^i(t))$ such that $F(\gamma^i(t)) = \gamma(t)$. Using the chain rule, we then have:

$$\gamma'(t) = \frac{\partial F}{\partial u_i}\frac{d\gamma^i}{dt}.$$

$\cancel{\gamma(t) = (\gamma^1(t), \cdots, \gamma^{n+1}(t))}$

$\gamma(t) = F(u_1(t), \cdots, u_n(t))$

$\gamma'(t) = \frac{\partial F}{\partial u_i}\frac{du_i}{dt}$

Recall that $|\gamma'(t)| = \sqrt{\langle\gamma'(t), \gamma'(t)\rangle}$ and that $\gamma'(t)$ lies on T_pM, we then have:

$$|\gamma'(t)| = \sqrt{g(\gamma'(t), \gamma'(t))}$$

We can then express it in terms of the matrix components g_{ij}'s:

(7.1) $$g(\gamma'(t), \gamma'(t)) = g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}$$

where g_{ij}'s are evaluated at the point $\gamma(t)$. Therefore, the arc-length can be expressed in terms of the first fundamental form by:

$$L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}\, dt = \int_a^b \sqrt{g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}}\, dt$$

Thm. Change of variable formula.

If $F(u_1, \cdots, u_n) : U \to M$ is a parametrization $\tilde{F}(v_1, \cdots, v_n)$ is a reparametrization. Then

$$\underbrace{\left[g_{v_i v_j} \right]}_{\frac{\partial \tilde{F}}{\partial v_i} \cdot \frac{\partial \tilde{F}}{\partial v_j}} = \left[\frac{\partial(v_1, \cdots, v_n)}{\partial(u_1, \cdots, u_n)}\right]^T \left[g_{u_i u_j} \right] \underbrace{\left[\frac{\partial(v_1, \cdots, v_n)}{\partial(u_1, \cdots, u_n)}\right]}_{\text{Jacobian}}$$

Proof: direct computation.

RMK. From this computation we see that the area form and arc length element is unchanged by reparametrization, since similar matrices have the same det.

Def. Isometry. A differeomorphism $f: M \longrightarrow N$ is called isometry if for any local parametrization F of M.
$\frac{\partial F}{\partial u_i} \cdot \frac{\partial F}{\partial u_j} = \frac{\partial (f\circ F)}{\partial u_i} \cdot \frac{\partial (f\circ F)}{\partial u_j}$ i.e. M and N has the same first fundamental form.

Two theorems are immidiate:

Thm If M, N isometric. $\gamma: (-\varepsilon, \varepsilon) \to M$ is a curve on M, then $L(\gamma) = L(f\circ\gamma)$ i.e. the length invariant under f.

Thm. If M, N isometric. $D: U \to M$ is a region on M, then $\text{Area}(D) = \text{Area}(f\circ D)$ i.e. the surface area is invariant under f.

RMK. • Although γ or D may be covered by more than 1 parametrizations, by summing up each piece, we may assume they are covered by a single chart.

• If $F: U \to M$ is a local chart for M, then $f\circ F: U \to N$ is a local chart for N since f is diffeomorphism.

area preserving: $\det(I_1) = \det(I_2)$. Conformal: preserves angle, also called isothermal.

~~PROPOSITION 4.4.2.~~ *Let $q: U \to M_1$ be a parametrization and $F: M_1 \to M_2$ a map. The map is an isometry if*

$F \circ q$ is a para. of M_2.

$$[\mathrm{I}_1] = [\mathrm{I}_2],$$

area preserving if

$$\det[\mathrm{I}_1] = \det[\mathrm{I}_2],$$

and conformal if

$$[\mathrm{I}_1] = \lambda^2 [\mathrm{I}_2]$$

for some non-zero function λ.

DEFINITION 4.4.3. In case the map is a parametrization $q: U \to M$ then we always use the *Cartesian metric* on U given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So the parametrization is an isometry or *Cartesian* when

$$[\mathrm{I}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

area preserving when

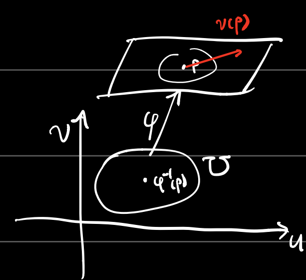
$$\det[\mathrm{I}] = 1,$$

and conformal or *isothermal* when

$$g_{uu} = g_{vv}, \quad [\mathrm{I}] = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} \quad \lambda \neq 0.$$
$$g_{uv} = 0.$$

Def. Vector field on S. Given regular surface S. A vector field on S is a diff. map $V: S \to \mathbb{R}^3$.

V is "normal" if $V(p) \perp T_pS$, $\forall p \in S$. "Unitary" if $|V(p)| = 1$.



$\varphi \circ v: U \to \mathbb{R}^3$ diff.

Lemma (local existence) If $\varphi: U \to S$ is a local chart

Then $\exists$ a unitary normal vector field on $\varphi(U)$. i.e. Every regular surface is locally orientable.

Proof. $\forall p \in \varphi(U)$: By def. of local chart. $\left\{ \frac{\partial \varphi}{\partial u}\Big|_{\varphi^{-1}(p)}, \frac{\partial \varphi}{\partial v}\Big|_{\varphi^{-1}(p)} \right\}$ is linearly independent.

$\Rightarrow N^{\varphi} = \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}\Big|_{\varphi^{-1}(p)}$ is the unitary normal vector at point $\varphi^{-1}(p)$. Warning: $N^{\varphi}: U \to \mathbb{R}^3$ is NOT a vector field on S, but on U

We change the domain of N^{φ}: $N(p) := N^{\varphi} \circ \varphi^{-1}(p) : S \to \mathbb{R}^3$ Done. ∎

Lemma (Global uniqueness) S connected. N_1, N_2 are both unitary normal vector fields on S. then

Either $N_1 \equiv N_2$, or $N_1 \equiv -N_2$ for all $p \in S$. i.e. it's impossible that $N_1(p) = N_2(p)$ but $N_1(q) = -N_2(q)$, for some $p, q \in S$.

Proof. $\forall p \in S$. $N_1(p), N_2(p) \perp T_pS$ So $N_1(p) = N_2(p)$ OR $N_1(p) + N_2(p) = 0$

Assume the first case: $N_1(p) = N_2(p)$ and assume $\exists q \in S$. $N_1(q) = -N_2(q)$. Then define $f: S \to \mathbb{R}^3$ by $f(p) = N_1(p) - N_2(p)$.

$g: S \to \mathbb{R}^3$ by $g(q) = N_1(q) + N_2(q)$. $A := \{ p \in S \mid f(p) = 0 \} = f^{-1}(0)$ and $B = g^{-1}(0)$ are closed disjoint subsets
by continuity of f, g

of S. By connectedness, either $A = \phi$ or $B = \phi$. ∎

Def. Orientability

DEFINITION 1. *A regular surface* S *is called* orientable *if it is possible to cover it with a family of coordinate neighborhoods in such a way that if a point* p ∈ S *belongs to two neighborhoods of this family, then the change of coordinates has positive Jacobian at* p. *The choice of such a family is called an* orientation *of* S, *and* S, *in this case, is called* oriented. *If such a choice is not possible, the surface is called* nonorientable.

Example 1) $S = \{ f^{-1}(a)$ where a is a regular point$\}$ $N(p) = \frac{\nabla f(p)}{|\nabla f(p)|}$ is a possible choice.

2) $S^2_{p_0}(r)$. $N(p) = \frac{p - p_0}{|p - p_0|}$ is a possible choice.

3). $S = \Gamma_f = \{ (x, y, f(x,y)) \}$. $N(p) = \frac{(1, 0, f_x) \times (0, 1, f_y)}{|(1, 0, f_x) \times (0, 1, f_y)|} = \frac{(f_x, f_y, -1)}{\sqrt{|\nabla f|^2 + 1}}$ is a possible choice.