Example 2. The sphere is an orientable surface. Instead of proceeding to a direct calculation, let us resort to a general argument. The sphere can be covered by two coordinate neighborhoods (using stereographic projection; see Exercise 16 of Sec. 2-2), with parameters (u, v) and $(\bar{u}, \bar{v})$, in such a way that the intersection W of these neighborhoods (the sphere minus two points) is a connected set. Fix a point p in W. If the Jacobian of the coordinate change at p is negative, we interchange u and v in the first system, and the Jacobian becomes positive. Since the Jacobian is different from zero in W and positive at $p \in W$, it follows from the connectedness of W that the Jacobian is everywhere positive. There exists, therefore, a family of coordinate neighborhoods satisfying Def. 1, and so the sphere is orientable.

By the argument just used, it is clear that *if a regular surface can be covered by two coordinate neighborhoods whose intersection is connected, then the surface is orientable.*

Before presenting an example of a nonorientable surface, we shall give a geometric interpretation of the idea of orientability of a regular surface in R^3.

As we have seen in Sec. 2-4, given a system of coordinates $\mathbf{x}(u, v)$ at p, we have a definite choice of a unit normal vector N at p by the rule

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(p). \tag{1}$$

Taking another system of local coordinates $\bar{\mathbf{x}}(\bar{u}, \bar{v})$ at p, we see that

$$\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}} = (\mathbf{x}_u \wedge \mathbf{x}_v)\frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})}, \tag{2}$$

where $\partial(u, v)/\partial(\bar{u}, \bar{v})$ is the Jacobian of the coordinate change. Hence, N will preserve its sign or change it, depending on whether $\partial(u, v)/\partial(\bar{u}, \bar{v})$ is positive or negative, respectively.

Thm. A regular surface S is "orientable" iff $\exists\, N: S \to S^2 \subseteq \mathbb{R}^3$ smooth normal field.

PROPOSITION 1. *A regular surface* $S \subset R^3$ *is orientable if and only if there exists a differentiable field of unit normal vectors* $N: S \longrightarrow R^3$ *on* S.

Proof. If S is orientable, it is possible to cover it with a family of coordinate neighborhoods so that, in the intersection of any two of them, the change of coordinates has a positive Jacobian. At the points $p = \mathbf{x}(u, v)$ of each neighborhood, we define $N(p) = N(u, v)$ by Eq. (1). $N(p)$ is well defined, since if p belongs to two coordinate neighborhoods, with parameters (u, v) and $(\bar{u}, \bar{v})$, the normal vector $N(u, v)$ and $N(\bar{u}, \bar{v})$ coincide by Eq. (2). Moreover, by Eq. (1), the coordinates of $N(u, v)$ in R^3 are differentiable functions of (u, v), and thus the mapping $N: S \longrightarrow R^3$ is differentiable, as desired.

On the other hand, let $N: S \to R^3$ be a differentiable field of unit normal vectors, and consider a family of *connected* coordinate neighborhoods covering S. For the points $p = \mathbf{x}(u, v)$ of each coordinate neighborhood $\mathbf{x}(U)$, $U \subset R^2$, it is possible, by the continuity of N and, if necessary, by interchanging u and v, to arrange that

$$N(p) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}.$$

In fact, the inner product

$$\left\langle N(p), \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} \right\rangle = f(p) = \pm 1$$

is a continuous function on $\mathbf{x}(U)$. Since $\mathbf{x}(U)$ is connected, the sign of f is constant. If $f = -1$, we interchange u and v in the parametrization, and the assertion follows.

Proceeding in this manner with all the coordinate neighborhoods, we have that in the intersection of any two of them, say, $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(\bar{u}, \bar{v})$, the Jacobian

$$\frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})}$$

is certainly positive; otherwise, we would have

$$\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = N(p) = -\frac{\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}}{|\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}|} = -N(p),$$

which is a contradiction. Hence, the given family of coordinate neighborhoods after undergoing certain interchanges of u and v satisfies the conditions of Def. 1, and S is, therefore, orientable. **Q.E.D.**

A non-example: Möbius Strip.

Example 3. We shall now describe an example of a nonorientable surface, the so-called *Möbius strip*. This surface is obtained (see Fig. 2-31) by considering the circle S^1 given by $x^2 + y^2 = 4$ and the open segment AB given in the yz plane by $y = 2$, $|z| < 1$. We move the center c of AB along S^1 and turn AB about c in the cz plane in such a manner that when c has passed through an angle u, AB has rotated by an angle $u/2$. When c completes one trip around the circle, AB returns to its initial position, with its end points inverted.

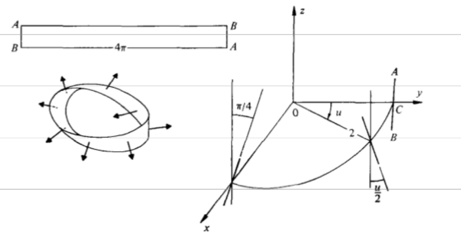


Figure 2-31

From the point of view of differentiability, it is as if we had identified the opposite (vertical) sides of a rectangle giving a twist to the rectangle so that each point of the side AB was identified with its symmetric point (Fig. 2-31).

It is geometrically evident that the Möbius strip M is a regular, nonorientable surface. In fact, if M were orientable, there would exist a differentiable field $N: M \to R^3$ of unit normal vectors. Taking these vectors along the circle $x^2 + y^2 = 4$ we see that after making one trip the vector N returns to its original position as $-N$, which is a contradiction.

We shall now give an analytic proof of the facts mentioned above.

A system of coordinates $\mathbf{x}: U \to M$ for the Möbius strip is given by

$$\mathbf{x}(u, v) = \left(\left(2 - v \sin\frac{u}{2}\right)\sin u, \left(2 - v \sin\frac{u}{2}\right)\cos u, v \cos\frac{u}{2}\right),$$

where $0 < u < 2\pi$ and $-1 < v < 1$. The corresponding coordinate neighborhood omits the points of the open interval $u = 0$. Then by taking the

origin of the u's at the x axis, we obtain another parametrization $\bar{\mathbf{x}}(\bar{u}, \bar{v})$ given by

$$x = \left\{2 - \bar{v}\sin\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right)\right\}\cos\bar{u},$$
$$y = -\left\{2 - \bar{v}\sin\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right)\right\}\sin\bar{u},$$
$$z = \bar{v}\cos\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right),$$

whose coordinate neighborhood omits the interval $u = \pi/2$. These two coordinate neighborhoods cover the Möbius strip and can be used to show that it is a regular surface.

Observe that the intersection of the two coordinate neighborhoods is not connected but consists of two connected components:

$$W_1 = \left\{x(u, v): \frac{\pi}{2} < u < 2\pi\right\},$$
$$W_2 = \left\{x(u, v): 0 < u < \frac{\pi}{2}\right\}.$$

The change of coordinates is given by

$$\left.\begin{aligned}\bar{u} &= u - \frac{\pi}{2}\\ \bar{v} &= v\end{aligned}\right\} \quad \text{in } W_1,$$

and

$$\left.\begin{aligned}\bar{u} &= \frac{3\pi}{2} + u\\ \bar{v} &= -v\end{aligned}\right\} \quad \text{in } W_2.$$

It follows that

$$\frac{\partial(\bar{u}, \bar{v})}{\partial(u, v)} = 1 > 0 \quad \text{in } W_1$$

and that

$$\frac{\partial(\bar{u}, \bar{v})}{\partial(u, v)} = -1 < 0 \quad \text{in } W_2.$$

To show that the Möbius strip is nonorientable, we suppose that it is possible to define a differentiable field of unit normal vectors $N: M \to R^3$. Interchanging u and v if necessary, we can assume that

$$N(p) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}$$

for any p in the coordinate neighborhood of $\mathbf{x}(u, v)$. Analogously, we may assume that

$$N(p) = \frac{\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}}{|\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}|}$$

at all points of the coordinate neighborhood of $\bar{\mathbf{x}}(\bar{u}, \bar{v})$. However, the Jacobian of the change of coordinates must be -1 in either W_1 or W_2 (depending on what changes of the type $u \to v$, $\bar{u} \to \bar{v}$ has to be made). If p is a point of that component of the intersection, then $N(p) = -N(p)$, which is a contradiction.

We have already seen that a surface which is the graph of a differentiable function is orientable. We shall now show that a surface which is the inverse image of a regular value of a differentiable function is also orientable. This is one of the reasons it is relatively difficult to construct examples of nonorientable, regular surfaces in R^3.

Question: here we only show that under this specific parametrization, it's impossible to find a smooth normal field But what about other possible parametrizations?

Answer: Although $N(p) = \frac{X_u \times X_v}{|X_u \times X_v|}$. it's independent of parametrization $X(u,v)$

A useful exercise. **5.** Let $\varphi: S_1 \longrightarrow S_2$ be a diffeomorphism.

a. Show that S_1 is orientable if and only if S_2 is orientable (*thus, orientability is preserved by diffeomorphisms*).

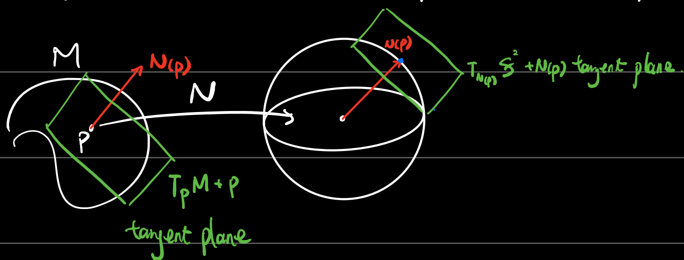
b. Let S_1 and S_2 be orientable and oriented. Prove that the diffeomorphism φ induces an orientation in S_2. Use the antipodal map of the sphere (Exercise 1, Sec. 2-3) to show that this orientation may be distinct (cf. Exercise 4) from the initial one (*thus, orientation itself may not be preserved by diffeomorphisms; note, however, that if* S_1 *and* S_2 *are connected, a diffeomorphism either preserves or "reverses" the orientation*).

$q \in S_2$: $n_2(q) = n_1(\varphi^{-1}(q))$ i.e. $n_2 = n_1 \circ \varphi^{-1}$. it depends on Jacobian of φ.

Guass map and second Fundamental form

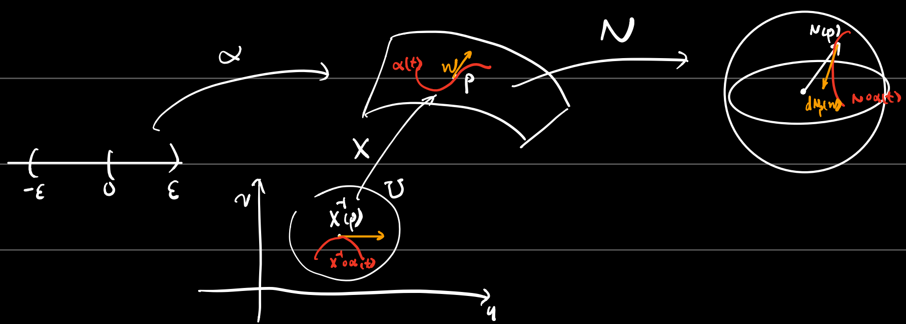
Def. Let $M \subseteq \mathbb{R}^3$ be a oriented regular surface. The map $N: M \to S^2$ is called the Guass map.

Since $dN_p: T_pM \longrightarrow T_{N(p)}S^2 \overset{\text{naturally}}{\cong} N(p)^\perp \cong T_pM \iff dN_p: T_pM \to T_pM$.



Thm. dN_p is a symmetric (self-adjoint) endomorphism on T_pM. i.e. $\langle dN_p(v), w\rangle = \langle v, dN_p(w)\rangle \quad \forall v, w \in T_pM$.

Proof. It suffice to check symmetric on a basis on T_pM. Consider $X(u,v): U \to X(U)$ a local parametrization around p. So $\{X_u, X_v\}$ is a basis of T_pM. $\alpha: (-\varepsilon, \varepsilon) \to M$. $\alpha(0) = p$ $\alpha'(0) = X_u$.



$dN_p(w) = \frac{d}{dt}\Big|_{t=0} (N \circ \alpha(t))$, ($w = \alpha'(0)$) write $\alpha(t) = X(u(t), v(t))$, $\alpha(0) = X(u(0), v(0)) = p$. $\alpha'(0) = X_u u'(0) + X_v v'(0)$

$= \frac{d}{dt}\Big|_{t=0} N(u(t), v(t))$

$\overset{\text{chain rule, linearity}}{=} u'(0)\frac{\partial N}{\partial u} + v'(0)\frac{\partial N}{\partial v}$ So in particular, $dN_p(X_u) = N_u$, $dN_p(X_v) = N_v$

Symmetric: $\langle N_u, X_v\rangle \overset{?}{=} \langle X_u, N_v\rangle$. By $N \perp T_pM$: $\begin{cases}\langle N, X_u\rangle = 0\\ \langle N, X_v\rangle = 0\end{cases}$ Differentiating: $\begin{cases}\frac{d}{dv}\langle N, X_u\rangle = 0 \quad ①\\ \frac{d}{du}\langle N, X_v\rangle = 0 \quad ②\end{cases}$

① $\langle N_v, X_u\rangle + \langle N, X_{uv}\rangle = 0$

② $\langle N_u, X_v\rangle + \langle N, X_{vu}\rangle = 0$ So dN_p is symmetric.

Done. ∎

Let's look at some examples.

Example 5. The method of the previous example, applied to the point $p = (0, 0, 0)$ of the paraboloid $z = x^2 + ky^2$, $k > 0$, shows that the unit vectors of the x axis and the y axis are eigenvectors of dN_p, with eigenvalues 2 and $2k$, respectively (assuming that N is pointing outwards from the region bounded by the paraboloid).

Parametrize $z = x^2 + ky^2$ by $X(u,v) = (u, v, u^2 + kv^2)$. Then $X_u = (1, 0, 2u)$ $v = (0, 1, 2ky)$

$$N(u,v) = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(-2u, -2kv, 1)}{\sqrt{4u^2 + 4k^2v^2 + 1}}$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2kv \end{vmatrix} = (+2u, -2kv, 1).$$

At $p = (0,0,0)$: $X_u(p) = (1,0,0)$ $X_v(p) = (0,1,0)$

$\therefore \alpha'(0) = (u'(0), v'(0), 0)$

$$N(t) = \frac{(+2u(t), -2kv(t), 1)}{\sqrt{4u^2 + 4k^2v^2 + 1}}, \quad N'(t) = \left[\frac{+2u'(t)\sqrt{4u^2 + 4k^2v^2 + 1} + 2u(t) \cdot \frac{8u \cdot u'(t) + 8k^2 v \cdot v'}{2\sqrt{\ }}}{4u^2 + 4k^2v^2 + 1}, \frac{-2kv'(t)\sqrt{\ } + 2kv \cdot \frac{8u \cdot u' + 8k^2 v \cdot v'(t)}{\sqrt{\ }}}{4u^2 + 4k^2v^2 + 1}, \frac{-\frac{1}{2}(8u \cdot u' + 8k^2 v \cdot v'(t))}{(4u^2 + 4k^2v^2 + 1)^{3/2}} \right]$$

$u(0) = 0 \quad v(0) = 0.$

$$\therefore N'(0) = \left(+2u'(0), -2kv'(0), 0 \right)$$

$$\Rightarrow dN_p(\underbrace{(u'(0), v'(0), 0)}_{\alpha'(0)}) = (+2u'(0), -2v'(0), 0) \text{ i.e. } dN_p\left(\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2a \\ -2k \\ 0 \end{bmatrix}.$$

$$dN_p = \begin{pmatrix} 2 & & \\ & -2k & \\ & & 0 \end{pmatrix} \text{ wrt. basis } \{X_u, X_v\}$$

Second Fundamental form $\mathrm{II}_p : T_pS \times T_pS \mapsto \mathbb{R}$ defined as

$\mathrm{II}_p(v, w) = -\langle dN_p(v), w \rangle.$

Matrix form: Suppose M has local chart $F : U \mapsto M$.

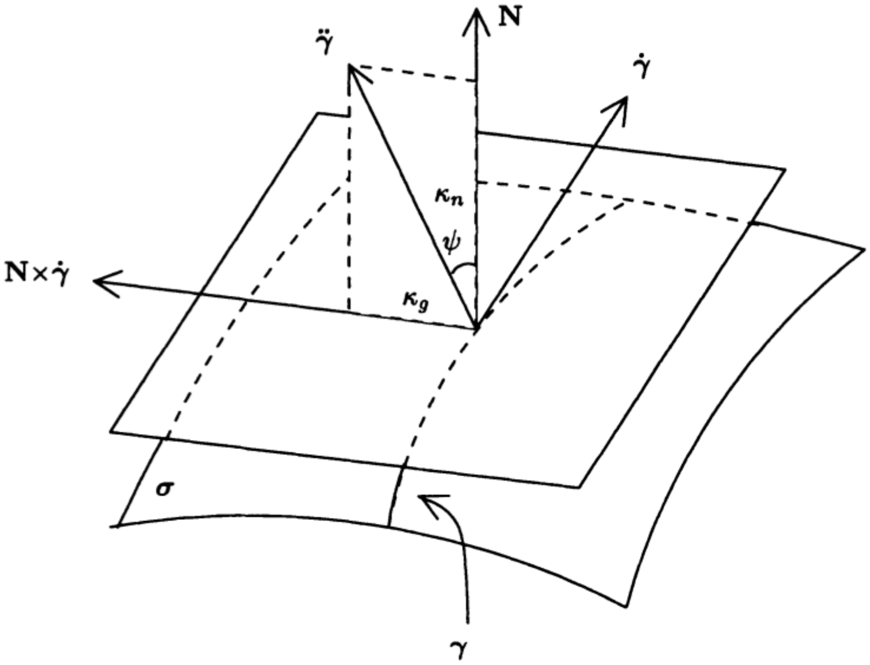
$\mathrm{II}_p(F_u, F_v) = -\langle dN_p(F_u), F_v \rangle = -\langle N_u, F_v \rangle = +\langle N, F_{uv} \rangle.$

$\mathrm{II}_p(F_u, F_u) = \langle N, F_{uu} \rangle$ $\mathrm{II}_p(F_v, F_v) = \langle N, F_{vv} \rangle.$

$$[\mathrm{II}] = \begin{bmatrix} N \cdot F_{uu} & N \cdot F_{uv} \\ N \cdot F_{uv} & N \cdot F_{vv} \end{bmatrix} \text{ real symmetric.}$$

$$\mathrm{II}_p(X, Y) = [x^u \ x^v][\mathrm{II}]\begin{bmatrix} Y^u \\ Y^v \end{bmatrix} \text{ where } X = [F_u \ F_v]\begin{bmatrix} x^u \\ x^v \end{bmatrix}$$

Geometry of II_p: Normal, Mean, Guassian, Geodesic curvatures.



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Consider a curve $\gamma(t) = F(u(t), v(t))$ on M. where F is a local chart. WLOG. suppose γ is arc-length parametrized. Then $k\vec{n} = \ddot{\gamma}$ where $\vec{n}$ is normal vector of γ.

of $\boldsymbol{\sigma}$, so $\dot{\boldsymbol{\gamma}}$, $\mathbf{N}$ and $\mathbf{N}\times\dot{\boldsymbol{\gamma}}$ are mutually perpendicular unit vectors. Again since $\boldsymbol{\gamma}$ is unit-speed, $\ddot{\boldsymbol{\gamma}}$ is perpendicular to $\dot{\boldsymbol{\gamma}}$, and hence is a linear combination of $\mathbf{N}$ and $\mathbf{N}\times\dot{\boldsymbol{\gamma}}$:

geodesic curvature.

$$\ddot{\boldsymbol{\gamma}} = \kappa_n\mathbf{N} + \kappa_g\mathbf{N}\times\dot{\boldsymbol{\gamma}}. \tag{5}$$

normal curvature.

then $k^2 = k_n^2 + k_g^2$.

Proposition 6.1

If $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ is a unit-speed curve on a surface patch $\boldsymbol{\sigma}$, its normal curvature is given by

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2,$$

where $Ldu^2 + 2Mdudv + Ndv^2$ is the second fundamental form of $\boldsymbol{\sigma}$.

This result means that two unit-speed curves passing through a point P on a surface and with the same tangent vector at P have the same normal curvature at P, since both κ_n and the tangent vector $\dot{\boldsymbol{\gamma}} = \boldsymbol{\sigma}_u\dot{u} + \boldsymbol{\sigma}_v\dot{v}$ depend only on $u, v, \dot{u}$ and $\dot{v}$ (and not on any higher derivatives of u and v).

Proof 6.1

We have, with $\mathbf{N}$ denoting the standard unit normal of $\boldsymbol{\sigma}$,

$$\begin{aligned}\kappa_n &= \mathbf{N}.\ddot{\boldsymbol{\gamma}} = \mathbf{N}.\frac{d}{dt}(\dot{\boldsymbol{\gamma}}) = \mathbf{N}.\frac{d}{dt}(\boldsymbol{\sigma}_u\dot{u} + \boldsymbol{\sigma}_v\dot{v}) \\ &= \mathbf{N}.(\boldsymbol{\sigma}_u\ddot{u} + \boldsymbol{\sigma}_v\ddot{v} + (\boldsymbol{\sigma}_{uu}\dot{u} + \boldsymbol{\sigma}_{uv}\dot{v})\dot{u} + (\boldsymbol{\sigma}_{uv}\dot{u} + \boldsymbol{\sigma}_{vv}\dot{v})\dot{v}) \\ &= L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2,\end{aligned}$$

$N \perp \sigma_u, \sigma_v$

■

Definition 6.1

The *principal curvatures* of a surface patch are the roots of the equation

$$\det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) = 0, \tag{11}$$

$\Leftrightarrow$ eigenvalue of $[I]^{-1}[II]$.

i.e.

$$\begin{vmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{vmatrix} = 0. \tag{12}$$

Proposition 6.3

Let κ_1 and κ_2 be the principal curvatures at a point P of a surface patch $\boldsymbol{\sigma}$. Then,

(i) κ_1 *and* κ_2 *are real numbers;*

(ii) if $\kappa_1 = \kappa_2 = \kappa$, *say, then* $\mathcal{F}_{II} = \kappa \mathcal{F}_I$ *and (hence) every tangent vector to* $\boldsymbol{\sigma}$ *at* P *is a principal vector;*

(iii) if $\kappa_1 \neq \kappa_2$, *then any two (non-zero) principal vectors* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *corresponding to* κ_1 *and* κ_2, *respectively, are perpendicular.*

In case (ii), P is called an *umbilic*.

Corollary 6.1 (Euler's Theorem)

Let $\boldsymbol{\gamma}$ *be a curve on a surface patch* $\boldsymbol{\sigma}$, *and let* κ_1 *and* κ_2 *be the principal curvatures of* $\boldsymbol{\sigma}$, *with non-zero principal vectors* $\mathbf{t}_1$ *and* $\mathbf{t}_2$. *Then, the normal curvature of* $\boldsymbol{\gamma}$ *is*

$$\kappa_n = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta,$$

where θ *is the angle between* $\dot{\boldsymbol{\gamma}}$ *and* $\mathbf{t}_1$.

Corollary 6.2

The principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point. Moreover, the principal vectors are the tangent vectors of the curves giving these maximum and minimum values.

Proposition 6.4

Let $\mathbf{N}$ *be the standard unit normal of a surface patch* $\boldsymbol{\sigma}(u, v)$. *Then,*

$$\mathbf{N}_u = a\boldsymbol{\sigma}_u + b\boldsymbol{\sigma}_v, \quad \mathbf{N}_v = c\boldsymbol{\sigma}_u + d\boldsymbol{\sigma}_v, \tag{20}$$

where

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_I^{-1}\mathcal{F}_{II}.$$

The matrix $\mathcal{F}_I^{-1}\mathcal{F}_{II}$ is called the *Weingarten matrix* of the surface patch $\boldsymbol{\sigma}$, and is denoted by $\mathcal{W}$.

Proposition 7.1

Let $\boldsymbol{\sigma}(u, v)$ *be a surface patch with first and second fundamental forms*

$$Edu^2 + 2Fdudv + Gdv^2 \quad \text{and} \quad Ldu^2 + 2Mdudv + Ndv^2,$$

respectively. Then,

$$[I] = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \qquad [II] = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

(i) $K = \frac{LN - M^2}{EG - F^2}$;

(ii) $H = \frac{LG - 2MF + NE}{2(EG - F^2)}$;

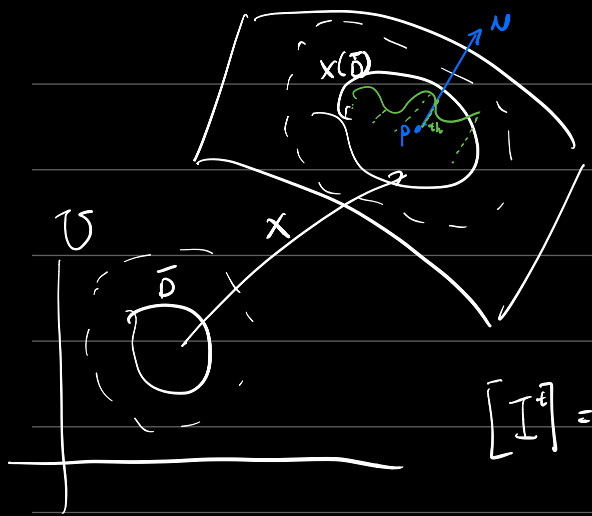
(iii) the principal curvatures are $H \pm \sqrt{H^2 - K}$.

Minimal surface.

Def. A regular surface is called "minimal" if $H \equiv 0$.

Consider $S \subseteq \mathbb{R}^3$. $X: U \to S$ is a local chart. $D \subset \bar{D} \subset U$ is a bounded region in U. $h: \bar{D} \to \mathbb{R}$ is a differentiable map.

The normal variation of $X(\bar{D})$ determined by h is the map $\varphi: \bar{D} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ $\varphi(u,v,t) = X(u,v) + t\,h(u,v)\,N(u,v)$
(a family of surfaces)



For $t \in (-\varepsilon, \varepsilon)$. Define $X^t = \varphi(\cdot\,\cdot\,t)$. Now

$$\begin{cases} X^t_u = X_u + t h_u N + t h N_u \\ X^t_v = X_v + t h_v N + t h N_v \end{cases}$$

$$[I^t] = \begin{bmatrix} E^t & F^t \\ F^t & G^t \end{bmatrix} \qquad \begin{aligned} E^t &= E + (2h\langle N_u, X_u\rangle)t + O(t^2) \\ F^t &= F + h(\langle X_u, N_v\rangle + \langle X_v, N_u\rangle)t + O(t^2) \\ G^t &= G + (2h\langle N_v, X_v\rangle)t + O(t^2) \end{aligned}$$

$[II] = \begin{bmatrix} e & f \\ f & g \end{bmatrix}$

$$\det(I_t) = E^t G^t - F^{t\,2} = EG - F^2 - 2h(Eg - 2Ff + Ge)t + O(t^2)$$

$$= (EG - F^2)(1 - 4hH) \quad \text{since } H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}$$

$$\text{Area of } X_t(\bar{D}) = \iint_{\bar{D}} \sqrt{\det(I_t)}\,du\,dv$$

$$= \iint (EG - F^2)^{\frac{1}{2}}(1 - 4hHt)^{\frac{1}{2}}\,du\,dv$$

$$\frac{d}{dt}\Big|_{t=0} \text{Area}(X(\bar{D})) = \frac{d}{dt}\Big|_{t=0} \iint (EG-F^2)^{\frac{1}{2}}(1-4hHt)^{\frac{1}{2}}\,du\,dv = \iint (EG-F^2)^{\frac{1}{2}} \frac{d}{dt}\Big|_{t=0} (1-4hHt)^{\frac{1}{2}}\,du\,dv$$

$$= -2\iint (EG-F^2)^{\frac{1}{2}} H \cdot h\,du\,dv \Leftarrow \text{1st variation of area.}$$

We want the surface "minimal". $X(\bar{D})$ minimal $\Rightarrow H \equiv 0 \Rightarrow \text{Area}(X(\bar{D}))$ is a critical point at $t=0$

Conversely, if 1st variation of area $= 0$ for all $h(u,v)$, then we must have $H \equiv 0$.

Warning: "minimal" just means it's a critical point of area function (with fixed boundary). it does not really means "minimum", which involves computing 2nd order derivative.

Geodesics

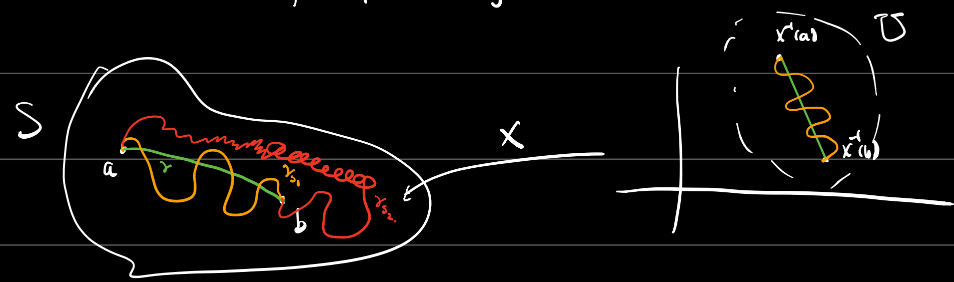
Def. A family of smooth curves on a regular surface S is a smooth map $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to S$ s.t.
$$(s,t) \longmapsto \Gamma(s,t)$$

- Fixing $s \in (-\varepsilon, \varepsilon)$. $\Gamma(s, \cdot) = \gamma_s(\cdot)$ is a smooth curve on S for $a \le t \le b$.

A smooth curve $\gamma : [a,b] \to S$ is a **geodesic** if for all family of curves $\{\gamma_s(t)\}_{-\varepsilon<s<\varepsilon}$ s.t.

- $\gamma_0(t) = \gamma(t)$
- $\gamma_s(a) = \gamma(a)$, $\gamma_s(b) = \gamma(b)$ $\forall s \in (-\varepsilon, \varepsilon)$ i.e. fixed end points.
- $\frac{d}{ds}\Big|_{s=0} \mathcal{L}(\gamma_s) = 0$. i.e. γ is a critical point of arc-length variation

length of γ_s



The following discussion is similar to least action principle in physics course

To get the equation of geodesics, it suffices to assume the target can be covered by a single chart.

Suppose $X : U \to S$ is a local chart that contains $\{\gamma_s\}$ $\gamma_s(t) = X(u(s,t), v(s,t))$

Let "$\dot{u}$", "$\dot{v}$" denote derivative w.r.t. t. R = norm squared of tangent vector of γ = $E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$

all are functions of s, t.

$$L(\gamma_s) = \int_a^b |\gamma_s'|\, dt = \int_a^b R^{\frac{1}{2}}\, dt$$

$$\Rightarrow \frac{d}{ds} L(\gamma_s) = \int_a^b \frac{d}{ds}\left(R^{\frac{1}{2}}\right) dt = \frac{1}{2}\int_a^b R^{-\frac{1}{2}} \frac{\partial R}{\partial s}\, dt$$

$$\frac{\partial R}{\partial s} = \left[\frac{\partial E}{\partial u}\dot{u}^2 + 2\frac{\partial F}{\partial u}\dot{u}\dot{v} + \frac{\partial G}{\partial u}\dot{v}^2\right]\frac{\partial u}{\partial s} + \left[\frac{\partial E}{\partial v}\dot{u}^2 + 2\frac{\partial F}{\partial v}\dot{u}\dot{v} + \frac{\partial G}{\partial v}\dot{v}^2\right]\frac{\partial v}{\partial s}$$

$$+ 2\left[E\dot{u} + F\dot{v}\right]\frac{\partial \dot{u}}{\partial s} + 2\left[F\dot{u} + G\dot{v}\right]\frac{\partial \dot{v}}{\partial s}$$

using boundary condition $\gamma_s(a) = \gamma(a)$ $\gamma_s(b) = \gamma(b)$ fixed, integration by parts gives

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \int_a^b P\frac{\partial u}{\partial s} + Q\frac{\partial v}{\partial s}\, dt \quad \text{where}$$

$$\begin{cases} P = \frac{1}{2}R^{-\frac{1}{2}}\left(\frac{\partial E}{\partial u}\dot{u}^2 + 2\frac{\partial F}{\partial u}\dot{u}\dot{v} + \frac{\partial G}{\partial u}\dot{v}^2\right) - \frac{\partial}{\partial t}\left(R^{-\frac{1}{2}}(E\dot{u} + F\dot{v})\right) = 0 \\ Q = \frac{1}{2}R^{-\frac{1}{2}}\left(\frac{\partial E}{\partial v}\dot{u}^2 + 2\frac{\partial F}{\partial v}\dot{u}\dot{v} + \frac{\partial G}{\partial v}\dot{v}^2\right) - \frac{\partial}{\partial t}\left(R^{-\frac{1}{2}}(F\dot{u} + G\dot{v})\right) = 0. \end{cases}$$

To get a geodesic, we must have $P \equiv 0$ $Q \equiv 0$ for all variations, similar in minimal surface.

Warning. geodesics does not guarantees the "minimum". It's just a critical point of arc-length.

As a Corollary. If $\gamma(s)$ is parametrized by arc-length, then

$$\gamma(s) \text{ is a geodesic} \iff \begin{cases} \frac{\partial}{\partial s}(E\dot{u} + F\dot{v}) = \frac{1}{2}\left(\frac{\partial E}{\partial u}\dot{u}^2 + 2\frac{\partial F}{\partial u}\dot{u}\dot{v} + \frac{\partial G}{\partial u}\dot{v}^2\right) \\ \frac{\partial}{\partial s}(F\dot{u} + G\dot{v}) = \frac{1}{2}\left(\frac{\partial E}{\partial v}\dot{u}^2 + 2\frac{\partial F}{\partial v}\dot{u}\dot{v} + \frac{\partial G}{\partial v}\dot{v}^2\right) \end{cases}$$

See "Notes on Tensor analysis" for geodesic eqs. using tensor notations

Corollary. Isometries preserve geodesics.

RMK. Geometric intepretation of geodesics: $\gamma(t)$ is geodesic iff. $k_n \equiv k(t)$, i.e. $k_g \equiv 0$. the normal of the surface parallel to the normal of the curve.

Proof. $\gamma(t) = X(u(t), v(t))$ $\quad \gamma' = \dot{u}X_u + \dot{v}X_v$ $\quad \gamma'' = \ddot{u}X_u + \ddot{v}X_v + \dot{u}\frac{d}{dt}X_u + \dot{v}\frac{d}{dt}X_v = \ddot{u}X_u + \ddot{v}X_v + (\dot{u}X_{uu} + \dot{v}X_{uv}) + (\dot{u}X_{vu} + \dot{v}X_{vv})$

We assume $\gamma(t)$ is arc-length parametrized.

$\langle \ddot{\gamma}, X_u \rangle = \ddot{u}E + \ddot{v}F + \dot{u}\langle X_{uu}, X_u\rangle + \dot{v}\langle X_{uv}, X_u\rangle + \dot{u}\langle X_{uv}, X_u\rangle + \dot{v}\langle X_{vv}, X_u\rangle$.

$E = \langle X_u, X_u\rangle \Rightarrow \frac{\partial E}{\partial u} = 2\langle X_{uu}, X_u\rangle$.

$= 0$, using equations of geodesics.

Similarly $\ddot{\gamma} \perp X_v$. $\Rightarrow \ddot{\gamma} \parallel N$. $\Rightarrow n \parallel N$. ∎

Thm. Given $\forall p \in M$, $\forall v \in T_pM$. $\exists!$ geodesic passing through p with tangent vector v at p.

Proof. It follows from Existence and Uniqueness Thm of sol. of ODEs. applying to geodesic Eqs.